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Logics of Space and Time

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Abstract

We review some literature in the field of spatial logics. The selection of papers we make is intended as an introductory guide in this broad area. In perspective, this review should be expanded in the future and become tailored to the use of spatial reasoning in the context of population models and their ODE / PDE approximations. The application to keep in mind is the analysis of population models where individuals are scattered over a spatial structure. In this context, typically, space is intended to be multi-dimensional, discrete or continuous; it may be useful to think in terms of Euclidean spaces, but also graph-based relational models may be the subject of spatial reasoning. Furthermore, metrics, measures, probabilities and rates may also be part of the requirements of an analysis methodology.

1 Introduction

Modal logics, model checking and static analysis enjoy an outstanding mathematical tradition, spanning over logics, abstract mathematics, artificial intelligence, theory of computation, system modelling, and optimisation. However, the *spatial* aspects of systems, that is, dealing with properties of entities that relate to their position, distance, connectivity, reachability in space, have never been truly emphasised in computer science. For the `QUANTICOL` project, it is important to be able to predicate over spatial aspects, and eventually find methods to certify that a given collective adaptive system satisfies specific requirements in this respect. A starting point is provided by so-called *spatial logics*, that have been studied from the point of view of (mostly modal) logics. The field of spatial logics is well developed in terms of descriptive languages and computability/complexity aspects. The development already started with early logicians such as Tarski, who studied possible semantics of classic modal logics, using topological spaces in place of frames. However, the frontier of current research does not yet reach verification problems, and in particular, discrete models are still a relatively unexplored field.

In this report, we study some relevant current literature dealing with continuous models, and start an analysis of the situation in the case of discrete structures. The interest for the `QUANTICOL` project in such an analysis comes from the conjecture that properties may be described using the same languages in the continuous, discrete, and relational (classical) case. This should provide an unifying view of temporal and spatial properties which is orthogonal to the kind of models that are taken into account. Our study is intended to be a starting point to understand which descriptive languages are interesting for the project. The next step, that we expect to be grounded on the results presented here, will be to cast the well known developments in spatio-temporal reasoning that we present, in the realm of discrete and finite structures, and to develop verification algorithms that are practical. This development constitutes a novel research line, that has not yet been explored. The lack of applications in the field of verification is also witnessed in the introduction to the book [1], that we use as our main reference.

Spatial logics predicate about entities that are related by a notion of *space*. The *possible worlds* of modal logics become complex mathematical structures. Very often, topological or metric spaces are used. Furthermore, a temporal dimension may be present, and the interplay of space and time gives rise to a rich design space, part of which is explored in this report. The existing literature is broad. The *Handbook of Spatial Logics* [1] constitutes an important classification and review effort. Most of the information contained in this report is based on the contents of the handbook and on references provided therein. Below, we summarise the most important design variables of a reasoning and verification framework based on spatial and temporal logics.

Spatial structures Space can be modelled as a discrete or continuous entity. This ought to be accommodated in a general setting by choosing appropriate abstract mathematical structures. *Topological spaces* are prototypical examples; however, one can obtain finer predicates by also introducing metrics (or costs, in the general case), therefore introducing *distance spaces* or *metric spaces*.

Spatial logics Spatial logics predicate on properties of entities located in the space; for example, one may be interested in entities that are *inside*, *outside* or on the *boundary* of regions of space where certain properties hold. Depending on the specific logical language, the entities described can be:

- Points in the space. In this case reasoning has a strongly local flavour. Global properties (e.g., a region of a space not having "holes") can not be expressed.
- Spaces. Global properties can easily be expressed if the point of view is shifted from the behaviour of an individual in a specified setting, to the analysis of several possible global scenarios consisting of all the entities in a given space.

- Regions of space. This approach combines reasoning on multiple entities simultaneously with a focus on the interaction (e.g., overlapping or contact) between areas having different properties.

Metrics and measures Distance-based logics extend topological logics. Formulas are indexed by intervals, which are used as constraints. Metric-topological properties are verified by a model if the topological part of the formula is verified, and the constraints are satisfied. For example, one may require that points satisfying a certain property are located at most at a specified distance from each other, or from points characterised by some other property.

Spatio-temporal logics The combination of spatial and temporal logics introduces more design variables, especially for what concerns the interplay between the spatial and the temporal component. Computational properties, such as decidability and complexity, of several possible combinations are examined in detail in [6]. We remark that for `QUANTICOL` such computational properties might prove less relevant than expected; see Remark 4.1.

Note 1.1. *This report does not cover other topics that are relevant for `QUANTICOL`. A partial list of well-known topics that are not covere here includes:* Metric Interval Temporal Logics*; logics of process calculi such as* MOSL, Ambient Logic*,* Pi-logic*, or* Separation Logic*. These subjects will be studied in related future work. With respect to ambient logics, we can already anticipate that the logic predicates on aspects of computation, that are somewhat orthogonal to metric or topologcal properties. These, in turn, are interesting features for `QUANTICOL`, dealt with in the logics presented in this report. Ambient logic is rather concerned with the idea of* named*, possibly* nested *locations, and, roughly, on information contained in locations being visible or hidden from other locations. The space where locations live does not have a physical structure; rather, it is an infinite set of names. These ideas may as well be interesting for the `QUANTICOL` project, but are not obviously amenable to be used as a replacement for topological and metric features of a spatially distributed population.*

2 Preliminaries

Here we introduce some relevant mathematical notions and facts that are used throughout the report. This section is intended as a minimal reference tailored to make the report self-contained, rather than as an introduction to modal logics, topology or other mathematical subjects, for which the reader is invited to consult more authoritative sources. The section on topology of [9] may be used as a gentle introduction; a comprehensive reference is [5]. For modal logics, a recommended read is [2].

2.1 Modal logics

We introduce the syntax of a basic modal logic, that we denote with $\mathcal{L}$, which forms the grounds for most other logics presented in this report.

Definition 2.1. Fix a set of *proposition letters* P. Let p denote an arbitrary letter. The syntax of $\mathcal{L}$ is described by the grammar:

$$\Phi ::= p \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Box\Phi \mid \Diamond\Phi$$

The relational semantics of $\mathcal{L}$ is given using *frames* and *models*.

Definition 2.2. Fixed a set P of *proposition letters*, a *frame* is a pair (X, R) of a set X and an *accessibility relation* $R \subseteq X \times X$. A *model* $\mathcal{M} = ((X, R), \mathcal{V})$ consists of a frame (X, R) and a valuation $\mathcal{V} : P \to \mathcal{P}(X)$, assigning to each proposition a set of points.

Truth of a formula is defined at a specific point $x \in X$.

Definition 2.3. Truth $\models$ of modal formulas in model $\mathcal{M} = ((X, R), \mathcal{V})$ at point $x \in X$ is defined by induction as follows:

$$\begin{array}{lcl}
\mathcal{M}, x \models \top & \iff & \textit{true} \\
\mathcal{M}, x \models \bot & \iff & \textit{false} \\
\mathcal{M}, x \models p & \iff & x \in \mathcal{V}(p) \\
\mathcal{M}, x \models \neg\phi & \iff & \texttt{not}\, \mathcal{M} \models \phi \\
\mathcal{M}, x \models \phi \wedge \psi & \iff & \mathcal{M}, x \models \phi \texttt{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \phi \vee \psi & \iff & \mathcal{M}, x \models \phi \texttt{ or } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \Box\phi & \iff & \forall y \in X.(x, y) \in R \implies \mathcal{M}, y \models \phi \\
\mathcal{M}, x \models \Diamond\phi & \iff & \exists y \in X.(x, y) \in R \wedge \mathcal{M}, y \models \phi
\end{array}$$

Formulas correspond to sets of points, as spelled out in the following definition.

Definition 2.4. For ϕ a formula, and $\mathcal{M}$ a model, we introduce the notation

$$\phi^{\mathcal{M}} \triangleq \{x \mid \mathcal{M}, x \models \phi\}$$

2.2 Temporal logics

Temporal logics are logics predicating on the behaviour of a system as time passes. These can be divided in two categories, those based on a *linear* notion of time, and those where time is *branching.*

2.2.1 Linear time

Logics dealing with time may have a *linear* flavour, that is, adopting a (typically countable) linear order $(X, \leq)$ intended as a set of *possible worlds* or time instants. A prototypical example is $\mathcal{LTL}$, which we define in this section.

Definition 2.5. The syntax of *linear temporal logics* $\mathcal{LTL}$ is given by

$$\Phi ::= p \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi\ \mathcal{U}\ \Phi$$

where p is a proposition letter drawn from a set P.

The semantics is given with respect to a model $((X, \leq), \mathcal{V})$ where $\mathcal{V} : P \to \mathcal{P}(X)$ assigns to a proposition the set of instants in which it is true.

Definition 2.6. The semantics of $\mathcal{LTL}$ in $\mathcal{M} = ((X, \leq), \mathcal{V})$ is given as follows:

$$\begin{array}{lcl}
\mathcal{M}, x \models \top & \iff & \textit{true} \\
\mathcal{M}, x \models \bot & \iff & \textit{false} \\
\mathcal{M}, x \models p & \iff & x \in \mathcal{V}(p) \\
\mathcal{M}, x \models \neg\phi & \iff & \texttt{not}\, \mathcal{M}, x \models \phi \\
\mathcal{M}, x \models \phi \wedge \psi & \iff & \mathcal{M}, x \models \phi \wedge \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \phi \vee \psi & \iff & \mathcal{M}, x \models \phi \vee \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \phi\ \mathcal{U}\ \psi & \iff & \exists y > x. \mathcal{M}, y \models \psi \wedge \forall z \in (x, y). \mathcal{M}, z \models \phi
\end{array}$$

Some interesting derived operators are:

- the *next step* modality $\bigcirc\phi \triangleq \bot\ \mathcal{U}\ \phi$, holding at $\mathcal{M}, x$ if and only if ϕ holds at $\mathcal{M}, x + 1$ (the successor of x in the linear order);
- the *sometime in the future* operator $\Diamond_F\phi \triangleq \top\ \mathcal{U}\ \phi$, asserting that there is a time when ϕ holds, and its dual *always in the future*, $\Box_F\phi \triangleq \neg(\Diamond_F\neg\phi)$;

- their non-strict variants $\Diamond_F^+\phi \triangleq \phi \vee \Diamond_F\phi$, and $\Box_F^+\phi \triangleq \phi \wedge \Box_F\phi$.

Remark 2.7. *The reader may notice a mismatch between the presentation of $\mathcal{LTL}$ that we give here (adapted from [6]) and classical presentations in the literature on program verification. Temporal logics are often used for reasoning about non-deterministic systems, with an initial state. Every trace (or computation) of a system with non-deterministic behaviour is a model as we defined it. However, the whole system is also referred to as a* model *in computer science literature, with the intended meaning that a system satisfies an $\mathcal{LTL}$ formula whenever all of its traces do.*

2.2.2 Branching time

Linear temporal logics predicate about behaviour of entities without making a distinction about potentially non-deterministic choices; events that are *possible* or *necessary* are collapsed into a deterministic view of a single future development.

Various extensions or variants of temporal logics exist in the literature, such as $\mathcal{CTL}$ or $\mathcal{CTL}^*$. In [6] the authors provide a simple introduction to the topic, by the means of an extension of $\mathcal{LTL}$ with operators E and A, denoting possibility and necessity, respectively.

Definition 2.8. The syntax of $\mathcal{BTL}$ is that of $\mathcal{LTL}$ from Definition 2.5, augmented with operators $\mathtt{E}\Phi$ and $\mathtt{A}\Phi$.

Models of $\mathcal{BTL}$ are based on rooted branching structures, that is, *trees*.

Definition 2.9. A *tree* is a structure $(X, <, r)$ where X is a set of points in time, $r \in X$ is the *root*, $<$ is an order, and for all $x \in X$ the structure $(\{y \in X \mid y < x\}, <)$ is a well-founded linear order. A history is a maximal linearly ordered suborder of $(X, <)$. An ω*-tree* is a tree where each history is isomorphic, as an order, to $(\mathbb{N}, <)$. Fixed a set P of proposition letters, a *branching model* is a structure $((X, <, r), \mathcal{H}, \mathcal{V})$ consisting of an ω-tree, a set of histories $\mathcal{H}$ (the possible flows of time) and a valuation $\mathcal{V} : P \to \mathcal{P}(X)$. Fixed a point x, let $\mathcal{H}(x) = \{h \in \mathcal{H} \mid x \in h\}$ be the set of histories *flowing through* x.

Fixed a model, truth is defined in terms of a history, and a point in time.

Definition 2.10. The semantics of $\mathcal{BTL}$ in $\mathcal{M} = ((X, <), \mathcal{V})$ at a history $h \in \mathcal{H}$ and point $x \in X$ is given as follows:

$$\begin{array}{lcl}
\mathcal{M}, h, x \models \top & \iff & \textit{true} \\
\mathcal{M}, h, x \models \bot & \iff & \textit{false} \\
\mathcal{M}, h, x \models p & \iff & x \in \mathcal{V}(p) \\
\mathcal{M}, h, x \models \neg\phi & \iff & \mathtt{not}\, \mathcal{M}, h, x \models \phi \\
\mathcal{M}, h, x \models \phi \wedge \psi & \iff & \mathcal{M}, h, x \models \phi \wedge \mathcal{M}, h, x \models \psi \\
\mathcal{M}, h, x \models \phi \vee \psi & \iff & \mathcal{M}, h, x \models \phi \vee \mathcal{M}, h, x \models \psi \\
\mathcal{M}, h, x \models \phi\, \mathcal{U}\, \psi & \iff & \exists y > x. \mathcal{M}, h, y \models \psi \wedge \forall z \in (x, y). \mathcal{M}, h, z \models \phi \\
\mathcal{M}, h, x \models \mathtt{E}\phi & \iff & \exists h' \in \mathcal{H}(x). \mathcal{M}, h', x \models \phi \\
\mathcal{M}, h, x \models \mathtt{A}\phi & \iff & \forall h' \in \mathcal{H}(x). \mathcal{M}, h', x \models \phi
\end{array}$$

2.3 Topological spaces

Definition 2.11. A *topological space* is a pair (X, O) of a set X and a collection $O \subseteq \mathcal{P}(X)$ of subsets of X called *open sets*, such that $\emptyset, X \in O$, and subject to closure under arbitrary unions and finite intersections.

Definition 2.12. An *open neighbourhood* of $x \in X$ is an open set o with $x \in o$.

Definition 2.13. A *continuous map* from (X_1, O_1) to (X_2, O_2) is a function $f : X_1 \to X_2$ such that for each $o \in O_2$, we have $f^{-1}(o) \in O_1$.

Definition 2.14. A *basis* of a topological space is a collection B of open sets such that every open set in O can be written as a union of elements of B.

Definition 2.15. A subset S of X is called *closed* if $X \setminus S \in O$. A *clopen* is a set that is both open and closed.

Definition 2.16. Given $S \subseteq X$, the *interior* of S, denoted by $\mathcal{I}(S)$, is the largest open set contained in S.

Definition 2.17. Given $S \subseteq X$, the *closure* of S, denoted by $\mathcal{C}(S)$, is the smallest closed set containing S.

The interior and closure are dual. Let $\overline{S}$ denote $X \setminus S$ (the complement of S in X). Then we have $\mathcal{I}(S) = \overline{\mathcal{C}(\overline{S})}$ and $\mathcal{C}(S) = \overline{\mathcal{I}(\overline{S})}$.

Definition 2.18. A *topological path* from $x \in X$ to $y \in X$ is a continuous map p from the closed interval $[0, 1] \subset \mathbb{R}$ to X such that $p(0) = x$ and $p(1) = y$.

Definition 2.19. A topological space (X, O) is *connected* whenever there is no choice of $o_1, o_2 \in O$ such that $o_1 \cup o_2 = X$, $o_1, o_2 \neq \emptyset$, $o_1 \cap o_2 = \emptyset$.

Note that, by definition, in a connected space, in order to cover the whole X, one needs to employ overlapping open sets.

2.4 Distance spaces and metric spaces

In this section we introduce *distance spaces* and *metric spaces*. The interested reader may refer to Section 3.1 of [6] to get some insight on distance spaces. In particular, qualitative notions such as "being at a short distance" can be modelled in distance spaces but not in metric spaces.

Definition 2.20. A *distance space* is pair (X, d) of a set X and a function $d : X \times X \to \mathbb{R}$ such that, for all $x, y \in X$, $d(x, y) = 0 \iff x = y$, and $d(x, y) \geq 0$.

Definition 2.21. A *metric space* is a distance space (X, d) such that, for all $x, y, z \in X$:

- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

Definition 2.22. A metric space can be equipped with the *metric topology* where the open sets are induced by the basis of *open balls*, that is, B is the collection of subsets $o \subseteq X$ such that there are $k \in \mathbb{R}$, $y \in X$ with $o = \{x \in X \mid d(x, y) < k\}$.

Definition 2.23. Given a metric space (X, d), $x, y \in X$, a subset $s \subseteq X$ is called a *metric segment* from x to y if there are a closed interval $[a, b] \subseteq \mathbb{R}$ and a distance-preserving map $p : [a, b] \to X$, with $p([a, b]) = s$, $p(a) = x$, $p(b) = y$.

Definition 2.24. Distances may be extended to sets. For $x \in X$, $S, T \subseteq X$, let $d(x, S) = \inf\{d(x, y) \mid y \in S\}$ and $d(S, T) = \inf\{d(x, y) \mid x \in S, y \in T\}$.

3 Modal logics of space

The material presented in this section mostly coomes from the book chapter [9], dealing with modal logics of space. The authors describe a number of important subjects in the area of topological spatial logics. First and foremost, the logics presented in this chapter are *modal*, that is, space is introduced through modal operators. Logics are also *purely spatial*, meaning that they deal with the spatial configuration of a system at a certain point in time, with no subsequent temporal evolution. We can roughly identify the following areas:

local topological logics: modalities identify *open sets* in which some or all points ought to satisfy a given property;

global topological logics: in addition, it is possible to predicate about the satisfaction of a certain property by classes of points in the space (e.g., *all* points);

epistemic logics: the spatial dimension is used to model the knowledge of agents;

geometric logics: truth depends upon the reciprocal location of geometrical objects;

distance logics: truth depends upon the distance between entities; these can either be expressed in precise terms (e.g., by values of a *metric*) or using relative distance (e.g. "x is closer to y than z"), again from a geometric point of view.

morphology: operations on space, such as *dilation* and *erosion*, that give foundations to *image processing*.

3.1 Topo-models and topo-logics

A topological space (Definition 2.11) may be used in place of a *frame* (see Definition 2.2) in order to interpret the modal logic $\mathcal{L}$ (Definition 2.1), obtaining *topological modal logics* or simply *topo-logics*. Recall $\mathcal{L}$ features boolean operations, and modal connectives $\Diamond$, $\Box$. We first need to define a topological *model*.

Definition 3.1. Fixed a set P of *proposition letters*, a *topological model* or *topo-model* $\mathcal{M} = ((X, O), \mathcal{V})$ consists of a topological space (X, O) and a valuation $\mathcal{V} : P \to \mathcal{P}(X)$, assigning to each proposition a set of points.

Truth of a formula is defined at a specific point x.

Definition 3.2. Truth $\models$ of modal formulas in model $\mathcal{M} = ((X, O), \mathcal{V})$ at point $x \in X$ is defined by induction as follows:

$$
\begin{array}{lll}
\mathcal{M}, x \models \top & \iff & \textit{true} \\
\mathcal{M}, x \models \bot & \iff & \textit{false} \\
\mathcal{M}, x \models p & \iff & x \in \mathcal{V}(p) \\
\mathcal{M}, x \models \neg\phi & \iff & \texttt{not}\, \mathcal{M} \models \phi \\
\mathcal{M}, x \models \phi \wedge \psi & \iff & \mathcal{M}, x \models \phi \;\texttt{and}\; \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \phi \vee \psi & \iff & \mathcal{M}, x \models \phi \;\texttt{or}\; \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \Box\phi & \iff & \exists o \in O.(x \in o \;\texttt{and}\; \forall y \in o. \mathcal{M}, y \models \phi) \\
\mathcal{M}, x \models \Diamond\phi & \iff & \forall o \in O.(x \in o \;\texttt{implies}\; \exists y \in o. \mathcal{M}, y \models \phi)
\end{array}
$$

The usual De Morgan-style dualities hold, including $\mathcal{M}, x \models \Box\phi \iff \mathcal{M}, x \models \neg\Diamond\neg\phi$. The interpretation of formulas identifies regions of space that depend on the valuation $\mathcal{V}$. In particular, note that the operation $\Box\phi$ identifies the topological *interior* of the region where ϕ holds. Dually, $\Diamond\phi$ denotes the topological *closure* of ϕ. An example formula which is widely used is the *boundary* of a property, which we introduce as a derived operator.

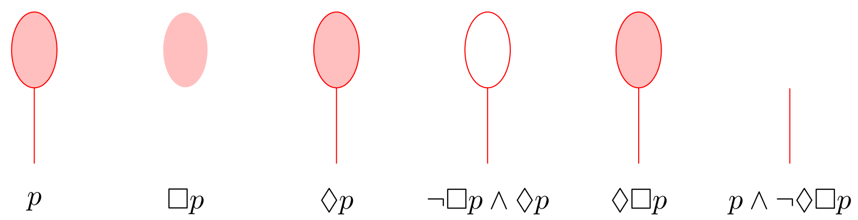

Figure 1: Topological interpretation of formulas over a topo-model.

Definition 3.3. The derived operator $\mathcal{B}\phi \triangleq \Diamond\phi \wedge \neg\Box\phi$ is called the *boundary* of ϕ.

Example 3.4. We report in Figure 1 the first example from [9]. The topological space here is the two-dimensional Euclidean plane $\mathbb{R}^2$ equipped with the metric topology. The only proposition letter is p and the valuation of p assigns to this property the shape of a "spoon" composed of a line segment and a filled ellipse. Various formulas can denote regions such as the boundary of the spoon, including or excluding the handle, the inner part of the spoon, the whole figure without the handle, etc.

3.2 Topo-bisimilarity and completeness

A natural question is what structures are logically equivalent, that is, how fine-grained is the logic. It turns out that logical equivalence coincides with the notion of *topological bisimilarity*. We detail the situation in this section. In the following, fix two models $\mathcal{M}_1 = ((X_1, O_1), \mathcal{V}_1)$ and $\mathcal{M}_2 = ((X_2, O_2), \mathcal{V}_2)$.

Definition 3.5. A *topological bisimulation*, or simply *topo-bisimulation*, is a relation $\mathcal{R} \subseteq X_1 \times X_2$ such that, for all $(x_1, x_2) \in \mathcal{R}$,

- For all $p \in P$, $x_1 \in \mathcal{V}_1(p)$ if and only if $x_2 \in \mathcal{V}_2(p)$, and
- for all $o_1 \in O_1$, whenever $x_1 \in o_1$, there is $o_2 \in O_2$ such that $x_2 \in o_2$ and for all $y_2 \in o_2$ there is $y_1 \in o_1$ with $(y_1, y_2) \in \mathcal{R}$, and
- for all $o_2 \in O_2$, whenever $x_2 \in o_2$, there is $o_1 \in O_1$ such that $x_1 \in o_1$ and for all $y_1 \in o_1$ there is $y_2 \in o_2$ with $(y_1, y_2) \in \mathcal{R}$.

Two points x_1, x_2 are *topo-bisimilar* if there is a topo-bisimulation relating them.

Bisimilarity equates points based on their local properties. Two points are bisimilar when, first of all, they have the same properties in their respective models. Then, it is required that, for every open set o_1 on one side, there is a choice of an open set o_2 on the other side, all points of which have a corresponding bisimilar point in o_1. This distinguishes points that are on the boundary of some property from points that are in its interior. Furthermore, the precise shape and size of properties in a model does not affect bisimilarity, which is only driven by the existence of open sets covering each property. See Example 3.6 for a graphical intuition.

Example 3.6. In Figure 2, we depict a topo-model using colours (red and blue) representing properties; we use small circles to denote points of interest in the space. In this example, the yellow points are all bisimilar, and so are the green points. The green points lay on the boundary of the blue property; the yellow points are in its interior. No green point is bisimilar to a yellow point. To see this, choose a yellow point x and a green point y. Note that there exist open sets containing x that are totally contained in the blue property. Let o be such an open set. For each choice of an open neighbourhood of y, on the other hand, there is a point outside the blue property, which does not have any corresponding point in o. This contradicts the first item in Definition 3.5.

Topo-logics have a strongly local flavour, and are not able to make a distinction between points, driven by properties that are at some distance from them. This is shown in Example 3.7.

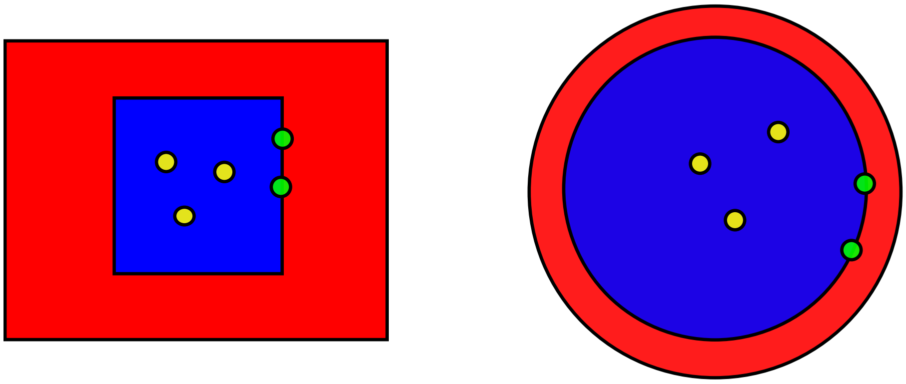

Figure 2: Bisimilar and non-bisimilar points under some predicates (represented by colours in the background). The yellow points are all bisimilar, and so are the green points; however, no green point is bisimilar to any yellow point.

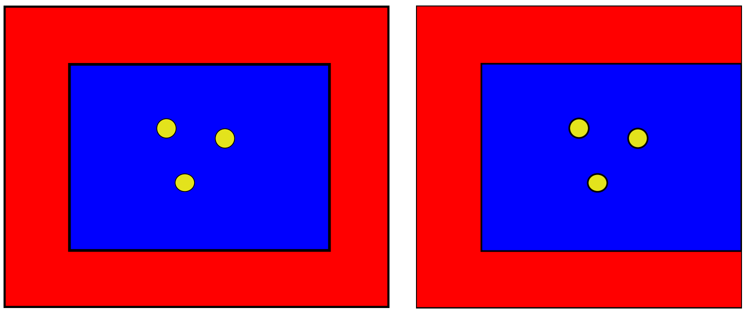

Figure 3: The yellow points (belonging to two different models) are all bisimilar, even if some of them are not completely surrounded by the red property.

Example 3.7. Consider Figure 3. The yellow points are all bisimilar, even though the red property completely surrounds only some of them. The red property can not affect bisimilarity of yellow points, since there is no physical contact between the two.

Topo-bisimilarity is the same relation as topo-logical equivalence. This is proved in [9], Theorems 5.4 and 5.5. We sum up the result as follows.

Theorem 3.8. *Two points $x_1 \in X_1$ and $x_2 \in X_2$ are topo-bisimilar if and only if they are* logically equivalent*, that is, for all formulas ϕ, it holds $\mathcal{M}_1, x_1 \models \phi$ if and only if $\mathcal{M}_2, x_2 \models \phi$.*

Example 3.9. An example is given in Figure 2. No yellow point is bisimilar to a green point. The distinguishing formula is $\neg\Box \mathit{blue} \wedge \Diamond \mathit{blue}$, that is, being on the boundary of the blue property.

A situation where Theorem 3.8 is useful is when one wants to prove that two models are logically equivalent. Then instead of verifying equivalence over all formulas (e.g., by induction), one can exhibit a topo-bisimulation.

3.3 Axiomatic aspects and relational semantics

From the point of view of logics, it is important to understand the axioms and the deductive power of a logic, and in particular its completeness with respect to classes of models. A logic is complete with respect to a class of models C, if all formulas that are true in every model are also provable using the axioms and rules of the logic. For such a statement to make sense in the setting of topo-logics, one needs to specify that a formula ϕ is true in a model $\mathcal{M} = ((X, O), \mathcal{V})$ if $\mathcal{M}, x \models \phi$ for all $x \in X$. Once this is established, various axioms are considered. As an example, we show those of the logic $\mathcal{S}4$, together with the relevant theorem. We refer the reader to [9] for more details.

Definition 3.10. The logic $\mathcal{S}4$ is $\mathcal{L}$ under the axioms K, T, 4.

$\Box(p \to q) \to (\Box p \to \Box q)$	(K)	distributivity
$\Box p \to \Box\Box p$	(4)	transitivity
$\Box p \to p$	(T)	reflexivity

These axioms further clarify the properties of topo-logics, as follows.

Theorem 3.11. *Assume the rules for* modus ponens *and* necessitation*:*

$$\frac{\phi \quad \phi \to \psi}{\psi} \qquad \frac{\phi}{\Box\phi}$$

The logic $\mathcal{S}4$ is complete with respect to topological models, that is, whenever ϕ is valid, it can be proved using the axioms $K, 4, T$, using modus ponens and necessitation.

Having seen this, and knowing that there are relational models of $\mathcal{S}4$, that is, the reflexive and transitive Kripke frames, one may wonder whether the connection is deeper. This is analysed in Section 2.4.1 of [9]. It is possible to derive a topological space from a frame, and the other way round, in a sound and complete way. The topological spaces that are used are *Alexandroff spaces*, where each point has a least open neighbourhood.

Remark 3.12. *The correspondence between topological spaces and frames is not easily extended to arbitrary frames, as transitivity and reflexivity always hold in topo-logics. On the other hand, requiring transitivity in all models may be a too limiting constraint, e.g., when "one-way" links are to be taken into account. Thus, it becomes interesting to further investigate non-transitive concepts of spatial models. See Section 5 for a starting point.*

3.4 Extended modal languages

Section 3.3 of [9] deals with some interesting extensions of the basic topo-logical framework. These include non-local reasoning (so-called "global properties") expressed through the use of *universal modalities*, temporal operators and logics that incorporate both spatial and temporal reasoning. We briefly review the first two ideas in Section 3.4.1 and Section 3.4.2. The interplay between space and time will be detailed in Section 4.

Remark 3.13. *Several other logics are considered in [9]. A non-exhaustive list includes epistemic logics, morphology, and logics of affine spaces, linear algebra and vector spaces, or metric geometry (see [9] for some literature). These subjects are not presented in great detail in this review, as establishing a link between them and the* `QUANTICOL` *project seems more remote, and may rather become a research question to be investigated separately.*

3.4.1 Global properties

The basic modal language $\mathcal{L}$ of Definition 2.1 can be extended with *existential* and *universal* operators, characterising points x where, respectively, a certain sub-formula holds at *some point* in the space where x lives, or rather it holds at *every point*. This adds a global flavor to reasoning, since one can predicate about the behaviour of all, or some, points of the space.

Definition 3.14. We introduce the logic $\mathcal{S}4_u$, having the same syntax as the logic $\mathcal{L}$ of Definition 2.1 with the two constructs $E\phi$ and $U\phi$, for ϕ a formula.

Definition 3.15. Given a topo-model $\mathcal{M} = ((X, O), \mathcal{V})$, the topological interpretation of the universal and existential modalities is given by:

$$\begin{array}{lcl} \mathcal{M}, x \models E\phi & \iff & \exists y \in X.\mathcal{M}, y \models \phi \\ \mathcal{M}, x \models U\phi & \iff & \forall y \in X.\mathcal{M}, y \models \phi \end{array}$$

It is noteworthy that the interpretation of formulas is still given *at some point* x. The obtained language expresses properties of *points* living in spaces that possess some point of interest. In the remainder of the section, [9] extends topo-bisimulations (Definition 3.5) to cater for global properties in Theorem 5.82, which extends Theorem 3.8. In particular, bisimulations are required to be *total* relations.

Theorem 3.16. *Topo-logic with semantics from Definition 3.2 is coarser than* $\mathcal{S}4_u$

Proof. We can use the example of Figure 3. The yellow points are all equivalent w.r.t. simple topological semantics, as they are bisimilar. However, consider the formula $\Phi \triangleq Ux.\phi(x) \rightarrow \psi(x)$, where $\phi(-)$ characterises the boundary of the blue property, and $\psi(-)$ characterises the boundary of the complement of the red property. Clearly Φ holds in the model on the left, and not in the one on the right. This shows that universal modalities are sensitive to the model where a point lays, rather than being just locally characterised. □

3.4.2 Temporal operators

Topological reasoning can borrow ideas from the realm of temporal logics. Section 3.3.2 of [9] considers the *until* operator.

Definition 3.17. Let $((X, O), \mathcal{V})$ be a topo-model. For $S \subseteq X$, recall that $\mathcal{C}(S)$ denotes the closure of s (Definition 2.17). The semantics of the until operator $\phi \ \mathcal{U} \ \psi$ is:

$$\begin{array}{lcl} \mathcal{M}, x \models \phi \ \mathcal{U} \ \psi & \iff & \exists o \in O.x \in o \wedge \\ & & \quad \forall y \in o.\mathcal{M}, y \models \phi \wedge \forall z \in \mathcal{C}(o) \cap \overline{o}.\mathcal{M}, z \models \psi \end{array}$$

In Definition 3.17, $\mathcal{C}(o) \cap \overline{o}$ coincides with the boundary of o; as o is open, it coincides with its interior. Therefore, the formula $\phi \ \mathcal{U} \ \psi$ is true at those points that are in the interior of ϕ, and for which there exists an open set having a boundary made up of points where ψ holds. The until operator gives to a logic a different expressive power from that of $\mathcal{S}4_u$, as shown in Example 3.18.

Example 3.18. Consider Figure 4, depicting a situation similar to Figure 3, in a single model. The logic $\mathcal{S}4_u$ is not able to tell green points apart from yellow points. This is because all these points live in the same model. Global formulas predicate on *all* the points of the space, or on the existence of *arbitrary* points satisfying a property, so they are not well-suited to distinguish points that are in the same model. However, the formula *blue* $\mathcal{U}$ *red* easily tells apart the yellow points, which are surrounded by the red property, at some distance, from the green points, that can reach the "outside" of the red property by open sets.

Logical reasoning becomes richer in logics with until. As an example, the following axiom holds:

$$p \ \mathcal{U} \ q \wedge r \ \mathcal{U} \ s \rightarrow (p \wedge r) \ \mathcal{U} \ (q \vee s)$$

The axiom becomes clear in the 2-dimensional Euclidean plane. For simplicity, let $p \triangleq r \triangleq \top$. Then the statement informally becomes: all points which are surrounded by two properties are also surrounded by their union.

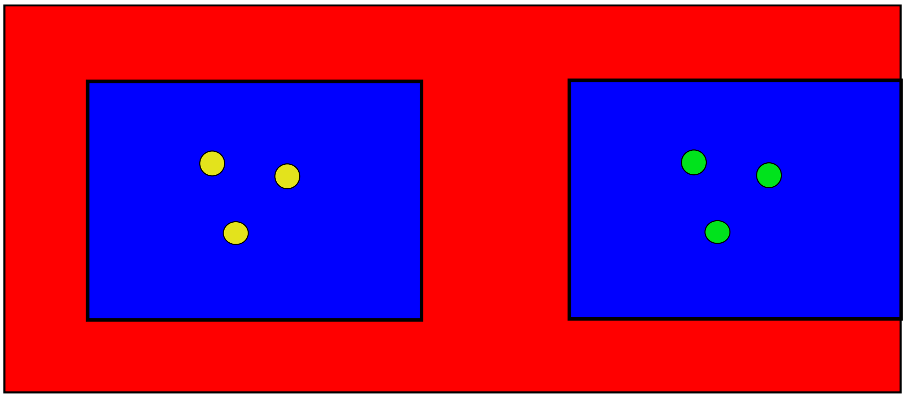

Figure 4: The green points and the yellow points are separated by the formula *blue* $\mathcal{U}$ *red*.

3.5 Logics of relative placement

Topo-logics reason on existence of open sets guaranteeing a certain property. Indeed, this is not the only way to reason about entities located in a spatial structure. One may be more concerned about the relative position of entities with respect to some reference point.

3.5.1 Affine geometry

In affine geometry it matters whether points are aligned to other points. A logical treatment of such situations can be done using a binary modality.

Definition 3.19. The syntax of logic $\mathcal{L}_B$ is given as follows:

$$\Phi ::= p \mid \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi \longleftrightarrow \Phi$$

In the logic, $\longleftrightarrow$ is a (binary) modal operator. Intuitively, a point satisfying $\phi \longleftrightarrow \psi$ must lay *between* a point satisfying ϕ and a point satisfying ψ. Formally, a model of the logic is a triple $(X, B, \mathcal{V})$, where B is a ternary relation expressing "betweenness" and $\mathcal{V}$ is a valuation from X into sets of propositions. It is intended that x is between y and z when $(y, x, z) \in B$.

Definition 3.20. Formulas of $\mathcal{L}_B$ are interpreted over models of the form $M = (X, B, \mathcal{V})$. Propositions and boolean connectives are interpreted as usual. Moreover, we have

$$M, x \models \phi \longleftrightarrow \psi \iff \exists y, z.(y, x, z) \in B \wedge M, y \models \phi \wedge M, z \models \psi$$

As typical in modal logics, logical equivalence is characterised by a notion of bisimilarity, spelled out as follows.

Definition 3.21. Given models $(X_1, B_1, \mathcal{V}_1)$ and $(X_2, B_2, \mathcal{V}_2)$, an *affine bisimulation* is a relation $\mathcal{R} \subseteq X_1 \times X_2$ such that, for all $(x_1, x_2) \in \mathcal{R}$:

- For all $p \in P$, $x_1 \in \mathcal{V}_1(p)$ if and only if $x_2 \in \mathcal{V}_2(p)$;
- for all $y_1, z_1 \in X_1$, $B(y_1, x_1, z_1) \implies \exists y_2, z_2 \in X_2.B(y_2, x_2, z_2) \wedge (y_1, y_2) \in \mathcal{R} \wedge (z_1, z_2) \in \mathcal{R}$;
- for all $y_2, z_2 \in X_2$, $B(y_2, x_2, z_2) \implies \exists y_1, z_1 \in X_1.B(y_1, x_1, z_1) \wedge (y_1, y_2) \in \mathcal{R} \wedge (z_1, z_2) \in \mathcal{R}$.

Two points x_1, x_2 are *topo-bisimilar* if there is a topo-bisimulation relating them.

3.5.2 Metric geometry

Besides affine geometry and betweenness, one may predicate on relative positioning by looking at the relative distance between points. For example, in a metric space, given points x, y, z, one may check whether $d(x,y) \leq d(x,z)$. This notion of being "closer than" may be exploited in a modal logic. This is done by adopting a binary modal operator $\leftleftarrows$, and enriching models with a "closer to" ternary predicate N. Intuitively, $(x,y,z) \in N$ whenever x is closer to y than to z. The formal definition is given below. Section 4.2.2 of [9] contains more details on the topic.

Definition 3.22. We let $M, x \models \phi \leftleftarrows \psi$ if and only if $\exists y, z. M, y \models \phi \wedge M, z \models \psi \wedge N(x,y,z)$.

Remark 3.23. *The interpretation of $\phi \leftleftarrows \psi$ is not completely intuitive. When x is closer to ϕ than to ψ, one could expect that there is y with $M, y \models \phi$ and* for all *z with $M, z \models \psi$ it should be the case that $(x,y,z) \in N$. Further investigation is required to understand these matters; there may be other semantics for the logic taking into account different interpretations of "being closer than". The semantics of the same operator in [6] appears to be different: see Definition 4.20.*

3.6 Morphology

Mathematical morphology studies the analysis and processing of geometrical structures. Subjects of study are sets of vectors in a vector space, that are intended to represent "images" in a generalised sense; e.g., a digital (monochromatic) image can be considered a set of vectors in $\mathbb{R}^2$. The theory is based on the idea that simple patterns can be composed in order to obtain more complex images. Therefore it makes sense to describe and process images using the patterns that constitute them.

As an example, we consider the *Minkowski addition* on two sets of vectors:

$$A \oplus B \triangleq \{a + b \mid a \in A \wedge b \in B\}$$

This operator is used to describe the "dilation" of an image A using a pattern B. Section 5 in [9] briefly sums up some developments that establish a link between morphology and logics. At the moment, we do not develop the topic further in this review; this is left as a possible future work whenever the topic becomes relevant for `QUANTICOL`.

4 Spatio-temporal reasoning

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The developments presented in this section mostly come from the book chapter [6], dealing with combinations of spatial and temporal logics, and from [7], dealing with so-called *dynamical topological logic*. In [6], the authors extend the basic framework of topo-logics that we discussed in Section 3 with simultaneous handling of the two dimensions of space, intended as a topological notion, and discrete (linear or branching) time, in the spirit of temporal logics. For this, first of all the various notions of spatial logics are analysed in detail. Thus, some aspects already presented in Section 3 are further clarified here. The paper explores various combinations of models and logics. The intent is to establish, for each examined configuration, algorithmic properties of the *satisfiability* problem, such as decidability or complexity.

Remark 4.1. *Decidability and complexity of the satisfiability problem is very important in logic, and also used in some verification techniques. However, there are verification methods that do not directly depend on satisfiability (e.g., "on the fly" model checking). Additionally, one should keep in mind that in `QUANTICOL` the focus may be on exploring approximate methods (e.g., mean field approximation, or statistical model checking), thus we are not considering satisfiability as a fundamental aspect in this review.*

4.1 Spatial models that evolve over time

In Section 3, we discussed models and logics of "static" space. The considered spatial structure does not evolve in time. In [6], the interplay between space and time is explored. In classical modal logics, the object of reasoning are *possible worlds* that may have observations on them. Spatio-temporal models enhance this point of view by equipping possible worlds with additional structure, such as a topology or a metric, that may change over time.

Spatial models can be classified as:

- based on *points* of a topological or metric space; a spatial object (say, the interpretation of a formula) coincides with the set of points it occupies.
- based on *regions*; in these models, spatial objects are *regions*, that is, portions of the space, subject to some uniformity or "well-behavedness" conditions (e.g., not containing or excluding isolated points).

The temporal aspects of the model can be dealt with as:

- *snapshot-based*; here the topological space in question evolves over time "as a whole"; in [6], time is modelled just as the linearly ordered natural numbers;
- *spatial transition systems* having a state space, a branching, possibly non-deterministic, transition relation, and associating to each state a spatial model; in this way, the *possible worlds* vision of state-based system is enhanced with the fact that each possible world is spatial;
- *dynamical systems* described by a topological space equipped with a total continuous map describing the step-wise dynamics of entities.

4.2 Topo-logics with global comparisons

The first logic presented in the paper extends topo-logics and the interpretation of Definition 3.2 by adding global operators that compare sets of elements described by spatial formulas. The presentation of [6] makes use of one comparison operator $\sqsubseteq$, denoting inclusion of terms. This is equivalent to adding to the logic so-called *universal modalities* (Definition 3.14) that predicate over all points of a model. The resulting logic is thus still called $\mathcal{S}4_u$. In this section, we redefine the logic, to match the notation used in [6] (see also Example 4.5).

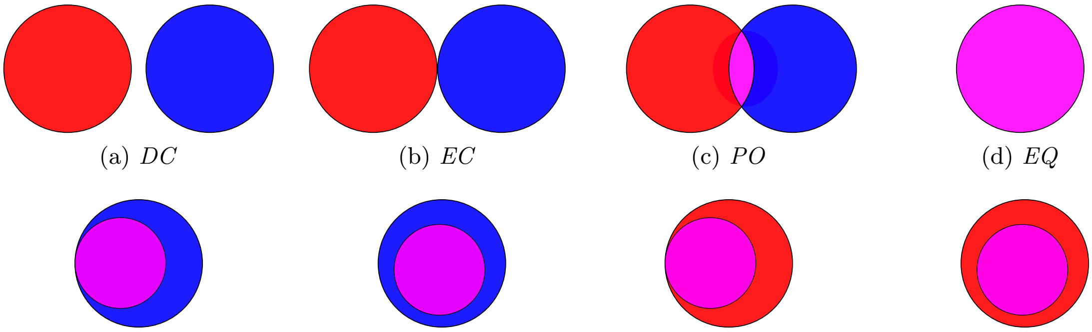
Definition 4.2. The syntax of $\mathcal{S}4_u$ is described by the grammar:

$$\begin{array}{lcl} \Phi & ::= & \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \tau \sqsubseteq \tau \\ \tau & ::= & p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid \tau \vee \tau \mid \Box\tau \mid \Diamond\tau \end{array}$$

The syntax is two-layered, including *formulas* Φ and *spatial terms* τ, which come from the syntax of $\mathcal{L}$ (see Definition 2.1). Truth of $\mathcal{S}4_u$ formulas is defined on models, not on their points (that is, formulas have a *global* interpretation).

Definition 4.3. For $\mathcal{M} = ((X, O), \mathcal{V})$ a topo-model, let $\mathcal{E}^{\mathcal{M}}(\tau)$ be the set $\{x \in X \mid \mathcal{M}, x \models \tau\}$, using Definition 3.2. Truth of formulas of $\mathcal{S}4_u$ is defined on models. We let $\mathcal{M} \models \tau_1 \sqsubseteq \tau_2$ if and only if $\mathcal{E}^{\mathcal{M}}(\tau_1) \subseteq \mathcal{E}^{\mathcal{M}}(\tau_2)$. Negation, conjunction and disjunction are interpreted as usual.

Note 4.4. *There is a conceptual mismatch between what we present in this section, and topo-logics with universal modalities of Section 3. In [6], satisfaction is defined on models, not points, and we have boolean operations also on formulas; in [9], these are not needed, as one still reasons on points, not models.*



(a) *DC* (b) *EC* (c) *PO* (d) *EQ*

(e) *TPP* (f) *NTPP* (g) TPP_i (h) $NTPP_i$

Figure 5: The eight $\mathcal{RCC}8$ operators; for each operator `bop`, a model is shown where `bop` (*red*, *blue*) is true; violet indicates the overlap of *red* and *blue*.

Example 4.5. The basic universal modalities of Definition 3.14 can be expressed in the language. For example $U\phi$ can be expressed as $\top \sqsubseteq \phi$. Conversely, $\phi \sqsubseteq \psi$ is equivalent to $U(\phi \rightarrow \psi)$.

We define abbreviations $(\phi = \psi) \triangleq \phi \sqsubseteq \psi \wedge \psi \sqsubseteq \phi$, and $\phi \neq \psi \triangleq \neg(\phi = \psi)$. The semantics of $\mathcal{S}4_u$ is powerful enough to express concepts such as density or connectedness. Below, recall that $\Box$ is interpreted as interior in topo-logics (Definition 3.2).

Example 4.6. The valuation of q is dense in the valuation of p (or more easily, q is dense in p) whenever

$$(q \sqsubseteq p) \wedge (p \sqsubseteq \Box q)$$

Consider as an example, $\mathcal{V}(q) = \mathbb{Q}$ and $\mathcal{V}(p) = \mathbb{R}$, in the real line.

4.3 Regular closed sets and calculi of regions

Several "non-well-behaved" sets are not relevant in everyday classical physics. For this reason, calculi of regions seek for good computational properties and simple semantics by predicating on restricted kinds of regions, typically the so-called *regular closed sets*.

Definition 4.7. A *regular closed set*, or *region*, in a topological space, is a set S such that $\mathcal{C}(\mathcal{I}(S)) = S$, where $\mathcal{C}$ and $\mathcal{I}$ come from Definition 2.17 and Definition 2.16.

On top of this definition of region, both linear and branching logics can be defined.

Definition 4.8. The syntax of the logic $\mathcal{RCC}8$ is given by:

$$\begin{array}{rcl} \Phi & ::= & \texttt{bop}\,(p, q) \\ \texttt{bop} & ::= & DC \mid EC \mid PO \mid EQ \mid TPP \mid NTPP \mid TPP_i \mid NTPP_i \end{array}$$

where p and q range over a set of proposition letters.

The semantics of the logic is easily understood by looking at Figure 5. The eight operators express various situations in which a region can be, with respect to another region. In particular, the operators denote that the first region is *disconnected* (a), *externally connected* (b), *partially overlapping* (c), *equal* (d), *tangential proper part* (e), *nontangential proper part* (f), *inverse of tangential proper part* (g), and *inverse of nontangential proper part* (h), respectively. In [6], the semantics mentioned above is formalised by translating $\mathcal{RCC}8$ into $\mathcal{S}4_u$.

Definition 4.9. Given a topo-model $((X, O), \mathcal{V})$, for each $p \in P$, let $\rho_p \triangleq \mathcal{CI}(\mathcal{V}(p))$. This ensures that propositions behave as $\mathcal{RCC}8$ region variables. The embedding of $\mathcal{RCC}8$ into $\mathcal{S}4_u$ is given below, for p and q ranging over proposition letters (we omit the "inverse" operators, whose translation is immediate):

$$\begin{array}{rcl} EC(p,q) & \triangleq & \neg(\rho_p \wedge \rho_q = \bot) \wedge (\Box\rho_p \wedge \Box\rho_q = \bot) \\ DC(p,q) & \triangleq & \rho_p \wedge \rho_q = \bot \\ EQ(p,q) & \triangleq & (\rho_p \sqsubseteq \rho_q) \wedge (\rho_q \sqsubseteq \rho_p) \\ PO(p,q) & \triangleq & \neg(\Box\rho_p \wedge \Box\rho_q = \bot) \wedge \neg(\rho_p \sqsubseteq \rho_q) \wedge \neg(\rho_q \sqsubseteq \rho_p) \\ TPP(p,q) & \triangleq & (\rho_p \sqsubseteq \rho_q) \wedge \neg(\rho_q \sqsubseteq \rho_p) \wedge \neg(\rho_p \sqsubseteq \Box\rho_q) \\ NTPP(p,q) & \triangleq & (\rho_p \sqsubseteq \Box\rho_q) \wedge \neg(\rho_q \sqsubseteq \rho_p) \end{array}$$

Remark 4.10. *In the encoding of NTPP, the part $\neg(\rho_q \sqsubseteq \rho_p)$ is not redundant. Consider the case when $\rho_p = \rho_q$ is an open set. This is not included in the definition of NTPP. In this case, $\rho_p \sqsubseteq \Box\rho_q$ holds; by looking at the semantics, we have $\rho_p = \rho_q = \mathcal{I}\rho_q$. Also, $\rho_q \sqsubseteq \rho_p$ holds.*

The embedding has strong properties, such as Theorem 4.11. Similar embeddings are possible also for the variants of $\mathcal{RCC}8$ analysed in [6]. By this, we are allowed to consider region calculi as fragments of $\mathcal{S}4_u$.

Theorem 4.11. *An $\mathcal{RCC}8$ formula is satisfiable in a topo-model if and only if its translation into $\mathcal{S}4_u$ is satisfiable in the same model.*

The logic $\mathcal{RCC}8$ may predicate on regions having boundaries or parts in common. However, one can easily find properties that are not expressible.

Example 4.12. The property "the three regions r, s, t have a point in common" can't be expressed in $\mathcal{RCC}8$.

A more expressive logic, although still unable to express the property of Example 4.12 can be obtained by introducing boolean operations on regions.

Definition 4.13. The logic $\mathcal{BRCC}8$ is the logic $\mathcal{RCC}8$ together with union, intersection and complementation operators on regions. Formally, the syntax is

$$\begin{array}{rcl} \Phi & ::= & \mathtt{bop}\,(\rho, \rho) \\ \rho & ::= & r \mid \top \mid \bot \mid \neg\rho \mid \rho \vee \rho \mid \rho \wedge \rho \end{array}$$

where `bop` is one of the region operators of $\mathcal{RCC}8$ (see Definition 4.8).

Remark 4.14. *Also $\mathcal{BRCC}8$ is embedded into $\mathcal{S}4_u$. Since not all boolean operations preserve regular open sets, the mapping of regions obtained by boolean operations is given in [6] using the closure of the interior of the obtained region. By this, in place of a boolean combination τ of $\mathcal{BRCC}8$ region terms, one can use the notation $\Diamond\Box\tau$, to denote an $\mathcal{S}4_u$ term which is in the image of the embedding.*

An even more general fragment of $\mathcal{S}4_u$ is $\mathcal{RC}$. This logic permits one to construct regions by arbitrary combinations of interior, boundary and exterior of each region in place of just boolean combinations of the regions themselves. This logic is finally able to express Example 4.12.

4.4 Logics of distance spaces

We now return to logics of arbitrary points, not regions, to deal with logics of distance spaces. When a set of points is equipped with a *distance* or a *metric*, as in Definition 2.20, logics may predicate on distances. In a generalised sense, such distances may be considered *costs* associated with moving from a point to another.

Remark 4.15. *In a metric space, one could also use $\mathcal{S}4$ and its variants, by interpreting them on the metric topology of the space. However, this is sub-optimal, as one could only reason on points that are "infinitely close" to other points (think of boundaries of a property).*

A simple way to define a logic in the spirit of topo-logics, with a metric flavour, is to start from $\mathcal{S}4_u$, replacing its existential and universal modalities with variants that are bounded by a distance.

Note 4.16. *In the definition of [6],* location constants *denoting specific points of the space are added to the logic. Such an addition is orthogonal to the distance operators, and may as well be done on $\mathcal{S}4_u$ and its fragments. Therefore, we omit location constants to keep the presentation clean.*

Definition 4.17. Fixed a set P of *proposition letters*, the syntax of the *logic of distance spaces $\mathcal{MS}$* is given by the grammar

$$\Phi ::= \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \tau \sqsubseteq \tau$$
$$\tau ::= p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid E^{=a}\tau \mid E^{<a}\tau \mid E^{<b}_{>a}\tau$$

where $p \in P$, and $a \in \mathbb{R}$.

The logic differs from $\mathcal{S}4_u$ as it has bounds on existential quantification (universals with bounds are obtained by duality). In [6] the models and semantics of the logic are not made explicit; this can be done as follows.

Definition 4.18. We let a *model* $((X, d), \mathcal{V}, \mathcal{U})$ consist of a metric space, equipped with a valuation for propositions $\mathcal{V} : P \to \mathcal{P}(X)$. The universal operator $\sqsubseteq$, boolean operations, and predicates, have the same semantics of Definition 4.3. The interpretation $\mathcal{E}^{\mathcal{M}}$ of terms in a model $\mathcal{M}$ is given below.

$$\begin{array}{rcl} \mathcal{E}^{\mathcal{M}}(E^{=a}\tau) & = & \{x \mid \exists y \in X.d(x, y) = a \wedge y \in \mathcal{E}^{\mathcal{M}}(\tau)\} \\ \mathcal{E}^{\mathcal{M}}(E^{<a}\tau) & = & \{x \mid \exists y \in X.d(x, y) < a \wedge y \in \mathcal{E}^{\mathcal{M}}(\tau)\} \\ \mathcal{E}^{\mathcal{M}}(E^{<b}_{>a}\tau) & = & \{x \mid \exists y \in X.d(x, y) \in (a, b) \wedge y \in \mathcal{E}^{\mathcal{M}}(\tau)\} \end{array}$$

Logic $\mathcal{MS}$ is undecidable; it is not even possible to decide whether there are valuations $\mathcal{V}$ and $\mathcal{U}$ that satisfy a formula in a given metric space. However, one may derive a weaker, decidable logic by removing the "doughnut" operator $E^{<b}_{>a}$. In that case, one can add also a $E^{\leq a}$ comparison operator, and then remove $E^{=}$, as it is derivable from the others.

4.4.1 Metric-topological reasoning

One may combine metric and topological reasoning by including the $\Diamond$ and $\Box$ operators from topo-logic into fragments of $\mathcal{MS}$. To keep decidability, the doughnut operator is omitted, and either one of the operators $E^{<a}$ or $E^{>a}$ is used, but not both, obtaining two different sub-logics; we illustrate one of them, named $\mathcal{MT}$.

Definition 4.19. The syntax of the logic $\mathcal{MT}$ is given by

$$\Phi ::= \top \mid \bot \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \tau \sqsubseteq \tau$$
$$\tau ::= p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid E^{<a}\tau \mid E^{\leq a}\tau \mid \Box\tau \mid \Diamond\tau$$

where $p \in P$, and $a \in \mathbb{R}$.

The models of $\mathcal{MT}$ could be, in principle, sets equipped with a topology and a metric. Moreover, it is possible to represent such structures as Kripke frames, in a faithful way with respect to the interpretation of formulas. Truth is defined using the metric for the distance operators, and the topology for the interior and closure operators. We omit the formal definition, which is just a merge of Definition 4.18 and Definition 3.2.

4.4.2 Relative distances

The "closer to" operator of Definition 3.22 is also studied in [6]. First of all, a basic logic is defined.

Definition 4.20. The logic $\mathcal{CSL}$ has the following syntax:

$$\Phi ::= \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \tau \sqsubseteq \tau$$
$$\tau ::= p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid \tau \vee \tau \mid \tau \leftleftharpoons \tau$$

where $p \in P$, and $a \in \mathbb{R}$.

Definition 4.21. The interpretation $\mathcal{E}^{\mathcal{M}}$ of region terms in a model $\mathcal{M} = ((X, d), \mathcal{V})$ space is as follows:

$$\begin{array}{lcllcl} \mathcal{E}^{\mathcal{M}}(p) & = & \mathcal{V}(p) & \mathcal{E}^{\mathcal{M}}(\neg\tau) & = & X \setminus \mathcal{E}^{\mathcal{M}}(\tau) \\ \mathcal{E}^{\mathcal{M}}(\tau_1 \wedge \tau_2) & = & \mathcal{E}^{\mathcal{M}}(\tau_1) \cap \mathcal{E}^{\mathcal{M}}(\tau_2) & \mathcal{E}^{\mathcal{M}}(\tau_1 \vee \tau_2) & = & \mathcal{E}^{\mathcal{M}}(\tau_1) \cup \mathcal{E}^{\mathcal{M}}(\tau_2) \\ \mathcal{E}^{\mathcal{M}}(\tau_1 \leftleftharpoons \tau_2) & = & \{x \in X \mid d(x, \mathcal{E}^{\mathcal{M}}(\tau_1)) < d(x, \mathcal{E}^{\mathcal{M}}(\tau_2))\} & & & \end{array}$$

Truth of formulas is as in the case of $\mathcal{S}4_u$ (Definition 4.3).

In Definition 4.21 we use the notation from Definition 2.4 to let $\phi^{\mathcal{M}}$ ($\psi^{\mathcal{M}}$, resp.) denote the set of points where ϕ (ψ) holds, and the notation from Definition 2.24 that extend distances to sets. Roughly, $\phi \leftleftharpoons \psi$ is true at x whenever x is closer to the set of points satisfying ϕ than to those satisfying ψ. We present some examples to appreciate the expressive power of the logic.

Example 4.22. We start by noticing that interior (and closure, by duality) is easily defined as $\Box\phi \triangleq \top \leftleftharpoons \neg\phi$. Also, the existential modality is obtained as $E\phi \triangleq \phi \leftleftharpoons \bot$.

Example 4.23. The term $(\neg(\phi \leftleftharpoons \psi) \wedge \neg(\psi \leftleftharpoons \phi))$ denotes the set of points that are equidistant from ϕ and ψ.

Adding more operators, such as location constants, gives rise to a very expressive logic. For example, [6] exhibits formulas determining the *Voronoi tessellation* of Euclidean spaces. The computational properties of such logics are mostly unknown.

4.5 Snapshot models

Section 5 of [6] introduces combinations of temporal and spatial logics. The considered variables are the logic chosen as the temporal fragment (such as, $\mathcal{LTL}$ or $\mathcal{BTL}$), the spatial fragment (such as, $\mathcal{S}4$ or $\mathcal{RCC}8$), and their interaction. The authors are mostly concerned about decidability and computational properties of the satisfiability problem. As per Remark 4.1, we review the section especially in the light of models and logics, rather than computational issues.

Example 4.24. One difficulty in combining spatial and temporal logics is that one can not just use the spatial and temporal components as if they were separated components. For example, let $\bigcirc\phi$ be the "next step" temporal operator. The formula $\bigcirc(EC(p,q)) \leftrightarrow EC(\bigcirc p, \bigcirc q)$ asserts that if two regions are externally connected in the next step, then the regions they will occupy in the next step are externally connected now. This seems to be a valid principle, but one could define spatio-temporal models in which it does not hold.

The assumption of [6] is that there is a fixed space, such as a topological or metric space, and the valuations of proposition letters vary over time. This is also what happens in the temporal models of most model checkers with state variables: variables values change over time and typically, literals represent propositions over (the values of) such variables.

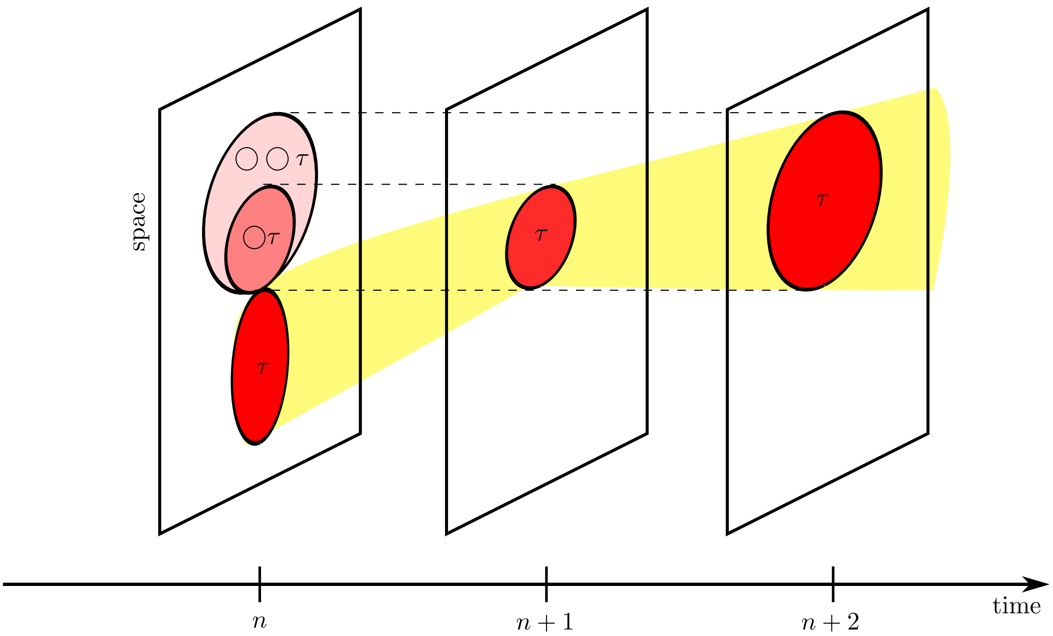


Figure 6: In spatio-temporal logics, a term interpreted at time n may refer to the interpretation of a subterm in some other instant of time.

Remark 4.25. *A key difference between most of the traditional temporal logics and spatio-temporal reasoning is the ability to predicate about the relationship between valuations of certain terms at different times. One may want to express that the position of a certain entity* now *is the same as the position of another entity* tomorrow*. In temporal logics, it is not typical to predicate on such properties; for example, it is rare to encounter logics so powerful to specify that the value of x today is the same as the value of y tomorrow; the situation is illustrated in Figure 6, where $\bigcirc\tau$ is a spatial term, whose semantics is the interpretation of term τ at the next instant of time. See also Example 4.24.*

Spatio-temporal models ought to be similar to spatial models. The valuation of proposition letters should be a function from *time* to functions that assign a set of points to each symbol. That is, models are collections of *snapshots* of the (spatial) state of affairs at a given time. In [6], the authors mostly analyse *discrete* flows of time. A temporal topo-model in this setting is in the form $((X, O), \mathcal{V}_i)$ where i belongs to a (possibly linear) order defining "time". Each i is a different instant. Instead of a topological space, one can also be based on a distance or metric space. This is made more precise in the following sections.

4.6 Spatio-temporal linear topo-logics

We now look at possible combination of temporal and spatial logics. A "maximalist" linear spatio-temporal logic combines $\mathcal{LTL}$ (Section 2.2.1) and $\mathcal{S}4_u$.

Definition 4.26. The logic $\mathcal{LTL} \times \mathcal{S}4_u$ has the following syntax:

$$\Phi ::= \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi\ \mathcal{U}\ \Phi \mid \tau \sqsubseteq \tau$$
$$\tau ::= p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid \tau \vee \tau \mid \Box\tau \mid \Diamond\tau \mid \tau\ \mathcal{U}\ \tau$$

The models of the logic are linear, discrete, spatial systems. The flow of time is modelled by natural numbers, whereas the domain for the valuation of propositions and spatial terms is a topological space. These models are referred to as *temporal topological models* or "tt-models" in [6].

Definition 4.27. Fixed a set P, a *tt-model* is a tuple $((X, O), \mathcal{V}_{n\in\mathbb{N}})$ where (X, O) is a topological space, and for each $n \in \mathbb{N}$, $\mathcal{V}_n : P \to \mathcal{P}(X)$ is the truth assignment of propositions.

The interpretation of terms τ and truth of formulas ϕ are defined below.

Definition 4.28. The interpretation of a term τ at time $n \in \mathbb{N}$ in model $\mathcal{M}$ is a subset $\mathcal{E}_n^{\mathcal{M}}(\tau)$ of X defined by induction as follows.

$$
\begin{array}{lclllcl}
\mathcal{E}_n^{\mathcal{M}}(p) & = & \mathcal{V}_n(p) & & \mathcal{E}_n^{\mathcal{M}}(\neg\tau) & = & X \setminus \mathcal{E}_n^{\mathcal{M}}(\tau) \\
\mathcal{E}_n^{\mathcal{M}}(\tau_1 \wedge \tau_2) & = & \mathcal{E}_n^{\mathcal{M}}(\tau_1) \cap \mathcal{E}_n^{\mathcal{M}}(\tau_2) & & \mathcal{E}_n^{\mathcal{M}}(\tau_1 \vee \tau_2) & = & \mathcal{E}_n^{\mathcal{M}}(\tau_1) \cup \mathcal{E}_n^{\mathcal{M}}(\tau_2) \\
\mathcal{E}_n^{\mathcal{M}}(\Box\tau) & = & \mathcal{I}(\mathcal{E}_n^{\mathcal{M}}(\tau)) & & \mathcal{E}_n^{\mathcal{M}}(\Diamond\tau) & = & \mathcal{C}(\mathcal{E}_n^{\mathcal{M}}(\tau)) \\
\mathcal{E}_n^{\mathcal{M}}(\tau_1\ \mathcal{U}\ \tau_2) & = & \bigcup_{m>n} \left(\mathcal{E}_m^{\mathcal{M}}(\tau_2) \cap \bigcap_{k\in(n,m)} \mathcal{E}_k^{\mathcal{M}}(\tau_1) \right) & & & &
\end{array}
$$

We are now ready to define truth of $\mathcal{LTL} \times \mathcal{S}4_u$.

Definition 4.29. Let $((X, O), \mathcal{V}_{n\in\mathbb{N}})$ be a tt-model. Truth of $\mathcal{LTL} \times \mathcal{S}4_u$ is defined in a model, for each point in time, as follows (we omit the boolean operations, whose interpretation is as usual):

$$
\begin{array}{lcl}
\mathcal{M}, n \models \tau_1 \sqsubseteq \tau_2 & \iff & \mathcal{V}_n(\tau_1) \subseteq \mathcal{V}_n(\tau_2) \\
\mathcal{M}, n \models \phi\ \mathcal{U}\ \psi & \iff & \exists m > n.\mathcal{M}, m \models \psi \wedge \forall k \in (n, m).\mathcal{M}, k \models \phi
\end{array}
$$

The logic $\mathcal{LTL} \times \mathcal{S}4_u$ is "maximal" in the sense that all possible constructs are included. In [6] the decidability properties of this logic and of combinations of $\mathcal{LTL}$ with the various fragments of $\mathcal{S}4_u$ are studied (these logics typically have very hard, if not undecidable, satisfiability problems).

It is noteworthy that the $\mathcal{U}$ operator of spatial terms, interpreted at time n, may refer to points in time different from n. This diverts from the spatial semantics of the until operator given in Definition 3.17, and it is non-typical in temporal logics, as already discussed informally in Remark 4.25.

Example 4.30. Consider the "next step" operator (see Section 2.2.1), defined as $\bigcirc\phi \triangleq \bot\ \mathcal{U}\ \phi$, and look at the semantics of this expression. The only value for m that does not falsify the terms of the outer union is $m = n + 1$, making the interval (n, m) empty. Thus, we have $\mathcal{E}_n^{\mathcal{M}} \bigcirc \phi = \mathcal{E}_{n+1}^{\mathcal{M}}\phi$, which defines a set of points that are "the points occupied by ϕ at the next instant of time". Recalling that $\mathcal{RCC}8$ is a fragment of $\mathcal{S}4_u$, consider the formula $EC(\Diamond\Box p, \bigcirc(\Diamond\Box q))$, asserting that the area that p occupies at the current time is externally connected to the area that q will occupy in the next instant.

4.7 Spatio-temporal branching topo-logics

If time is not linear, we can combine logics such as $\mathcal{BTL}$ (see Section 2.2.2) with spatial operators. In [6], the authors investigate combinations with operators from region calculi. First of all, we define models, where we now need to make explicit the order representing time (in linear spatio-temporal logics, the order is implicitly assumed to coincide with the natural numbers).

Definition 4.31. Fixed a set P of proposition letters, a *branching-time topological model*, or *btt-model*, is a tuple $\mathcal{M} = ((N, <), \mathcal{H}, (X, O), \mathcal{V}_{n\in N})$ where $(N, <)$ is an ω-tree (Definition 2.9), $\mathcal{H}$ is a set of histories, (X, O) is a topological space, and $\mathcal{V}$ is a family, indexed by N, of functions, so that for $n \in N$, $\mathcal{V}_n : P \to \mathcal{P}(X)$ is a map associating to each proposition letter the set of points in which it holds at time n.

Remarkably, the valuation of propositions does not depend on the actual history.

The branching spatio-temporal logics analysed in [6] can be of two flavours, either allowing operators `E` and `A` to be applied just to formulas, or to region terms, of the spatial fragment. The first logic is called $\mathcal{BTL} \times \mathcal{BRCC}8$; the second one $\mathcal{BTL} \times \mathcal{BRCC}8_x$.

Example 4.32. The formula $\mathtt{E}\bigcirc(PO(p,q))$ asserts that it is possible that regions denoted by p and q will be partially overlapping in the next instant of time. This formula belongs to $\mathcal{BTL} \times \mathcal{BRCC}8$. Consider, instead, the formula $\bigcirc PO(\mathtt{E}p, \mathtt{A}q)$, in $\mathcal{BTL} \times \mathcal{BRCC}8_x$. This describes a situation where, in the next instant of time (in the current history), the regions of points that p will possibly occupy overlaps with the region of those points that q will necessarily cover.

Taking advantage of the embedding of $\mathcal{RCC}8$ and its variants into $\mathcal{S}4_u$ (see Definition 4.9), we may let region terms be in the form $\Diamond\Box\tau$, for τ a $\mathcal{S}4_u$ term (see Remark 4.14). In the following, we shall provide a semantics for spatial terms, and formulas, of $\mathcal{BTL} \times \mathcal{BRCC}8_x$, as $\mathcal{BTL} \times \mathcal{BRCC}8$ is a fragment of it.

Definition 4.33. Fixed a btt-model $\mathcal{M}$ as in Definition 4.31, the interpretation of region terms of $\mathcal{BTL} \times \mathcal{BRCC}8_x$ is a map $\mathcal{E}^{\mathcal{M}}_{h,n}$ denoting the points occupied by a term at time n in the history h, defined by induction on terms as follows:

$$
\begin{array}{lcl}
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box p) & = & \mathcal{C}(\mathcal{I}(\mathcal{V}_n(p))) \\
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box\neg\tau) & = & \mathcal{C}(\mathcal{I}(X \setminus \mathcal{E}^{\mathcal{M}}_{h,n}(\tau))) \\
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box\tau_1 \wedge \tau_2) & = & \mathcal{C}(\mathcal{I}(\mathcal{E}^{\mathcal{M}}_{h,n}(\tau_1) \cap \mathcal{E}^{\mathcal{M}}_{h,n}(\tau_2))) \\
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box\tau_1 \vee \tau_2) & = & \mathcal{C}(\mathcal{I}(\mathcal{E}^{\mathcal{M}}_{h,n}(\tau_1) \cup \mathcal{E}^{\mathcal{M}}_{h,n}(\tau_2))) \\
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box\tau_1\ \mathcal{U}\ \tau_2) & = & \mathcal{C}\left(\mathcal{I}\left(\bigcup_{r\in h, m>n}(\mathcal{E}^{\mathcal{M}}_{h,m}(\tau_2) \cap \bigcap_{k\in(n,m)} \mathcal{E}^{\mathcal{M}}_{h,k}(\tau_1))\right)\right) \\
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box\mathtt{E}\tau) & = & \mathcal{C}\left(\mathcal{I}\left(\bigcup_{h'\in\mathcal{H}(n)} \mathcal{E}^{\mathcal{M}}_{h',n}(\tau)\right)\right) \\
\mathcal{E}^{\mathcal{M}}_{h,n}(\Diamond\Box\mathtt{A}\tau) & = & \mathcal{C}\left(\mathcal{I}\left(\bigcap_{h'\in\mathcal{H}(n)} \mathcal{E}^{\mathcal{M}}_{h',n}(\tau)\right)\right)
\end{array}
$$

Finally, we can define the truth value of formulas.

Definition 4.34. Truth of $\mathcal{BTL} \times \mathcal{BRCC}8_x$ formulas is defined by induction in a btt-model $\mathcal{M} = ((N,<), \mathcal{H}, (X,O), \mathcal{V}_{n\in N})$, at history h, and time n, by the following inductive clauses, where `bop` denotes one of the $\mathcal{RCC}8$ region operators:

$$
\begin{array}{lcl}
\mathcal{M},h,n \models \mathtt{bop}\,(\tau_1,\tau_2) & \iff & (X,O) \models \mathtt{bop}\,(\mathcal{E}^{\mathcal{M}}_{h,n}(\tau_1), \mathcal{E}^{\mathcal{M}}_{h,n}(\tau_2)) \\
\mathcal{M},h,n \models \top & \iff & \textit{true} \\
\mathcal{M},h,n \models \bot & \iff & \textit{false} \\
\mathcal{M},h,n \models \neg\phi & \iff & \mathtt{not}\,\mathcal{M},h,n \models \phi \\
\mathcal{M},h,n \models \phi\wedge\psi & \iff & \mathcal{M},h,n \models \phi \wedge \mathcal{M},h,n \models \psi \\
\mathcal{M},h,n \models \phi\vee\psi & \iff & \mathcal{M},h,n \models \phi \vee \mathcal{M},h,n \models \psi \\
\mathcal{M},h,n \models \phi\ \mathcal{U}\ \psi & \iff & \exists m \in h.m > n \wedge \\
 & & \quad \mathcal{M},h,m \models \psi \wedge \forall k \in (n,m).\mathcal{M},h,k \models \phi \\
\mathcal{M},h,n \models \mathtt{E}\phi & \iff & \exists h' \in \mathcal{H}(n).\mathcal{M},h',n \models \phi \\
\mathcal{M},h,n \models \mathtt{A}\phi & \iff & \forall h' \in \mathcal{H}(n).\mathcal{M},h',n \models \phi
\end{array}
$$

4.8 Spatio-temporal distance logics

As explained in [6], not much is known about temporal extensions of distance logics, especially from the point of view of computational properties; in general, the satisfiability problem is not tractable.

Definition 4.35. The syntax of the logic $\mathcal{LTL} \times \mathcal{MS}^{\leq}$ is given by the following grammar:

$$\Phi ::= \top \mid \bot \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi\ \mathcal{U}\ \Phi \mid \tau \sqsubseteq \tau$$
$$\tau ::= p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid \tau \vee \tau \mid E^{\leq a}\tau \mid \tau\ \mathcal{U}\ \tau$$

Models of the logic combine distance spaces with valuations that depend on time.

Definition 4.36. A *metric temporal model*, or *mt-model*, is a structure of the form $((X, d), \mathcal{V}_{\_})$, where for all time instants $n \in \mathbb{N}$, $\mathcal{V}_n : P \to \mathcal{P}(X)$.

Valuations and truth are defined as expected.

Definition 4.37. The interpretation of region terms is a function $\mathcal{E}_n^{\mathcal{M}}$, defined for each mt-model $\mathcal{M} = ((X, d), \mathcal{V}_{\_})$ and time instant n. We let

$$\begin{array}{lcllcl}
\mathcal{E}_n^{\mathcal{M}}(p) & = & \mathcal{V}_n(p) & \mathcal{E}_n^{\mathcal{M}}(\neg\tau) & = & X \setminus \mathcal{E}_n^{\mathcal{M}}(\tau) \\
\mathcal{E}_n^{\mathcal{M}}(\tau_1 \wedge \tau_2) & = & \mathcal{E}_n^{\mathcal{M}}(\tau_1) \cap \mathcal{E}_n^{\mathcal{M}}(\tau_2) & \mathcal{E}_n^{\mathcal{M}}(\tau_1 \vee \tau_2) & = & \mathcal{E}_n^{\mathcal{M}}(\tau_1) \cup \mathcal{E}_n^{\mathcal{M}}(\tau_2) \\
\mathcal{E}_n^{\mathcal{M}}(E^{\leq a}\tau) & = & \{x \in X \mid \exists y. d(x, y) \leq a \wedge y \in \mathcal{E}_n^{\mathcal{M}}(\tau)\} & & & \\
\mathcal{E}_n^{\mathcal{M}}(\tau_1\ \mathcal{U}\ \tau_2) & = & \bigcup_{m>n} \left(\mathcal{E}_m^{\mathcal{M}}(\tau_2) \cap \bigcap_{k \in (n,m)} \mathcal{E}_k^{\mathcal{M}}(\tau_1) \right) & & &
\end{array}$$

Finally, truth is defined as in $\mathcal{LTL} \times \mathcal{S}4_u$ (we omit the definition, as does [6]).

4.9 Logics of dynamical systems

Finally, [6] deals with logics of *dynamical systems*. This subject diverts from the other logics that we discussed so far, as models are in the form of a vector space, and a function that expresses the *dynamics* of the system.

Definition 4.38. A *dynamical model* is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a spatial model based on a set of points X, such as, a topo-model or a metric model, whereas $g : X \to X$ is a total function.

When the spatial model is a topo-model Often g has additional constraints, such as various forms of continuity. In this setting, one typically investigates the *orbit* of a point x, that is, the set $\mathcal{O}_g(x) = \{g^i(x) \mid i \in \mathbb{N}\}$, or the orbits of several points at once. A typical question is whether, fixed a set of points, all of their orbits will eventually reach and stay forever into some region of the space, possibly, avoiding other regions.

Dynamical systems are a well-known, and widely used mathematical model. In [6], two examples are presented. The first one describes a continuous system containing a physical body of mass m. The laws of physics provide a system of two differential equations whose solution governs the motion of the body, starting from a given position and speed. The second example is a discrete system, namely Conway's *game of life*, which is seen as a dynamical system by letting g be the transition function of each point in the grid of the game. It is worth looking at this example in more detail, especially because of the topology imposed on the state space of the resulting system.

Example 4.39. The *game of life* consists of the infinite grid $\mathbb{Z} \times \mathbb{Z}$ equipped with a boolean colouring in each cell, so that each cell can be either *occupied* or *vacant*, and an *update rule*, described as follows. At each (discrete) step of time, for each cell, if the cell is vacant, and has three occupied neighbours, it becomes occupied. If the cell is occupied, and two or three neighbours are occupied, the cell stays occupied. In all other cases, the cell becomes vacant.

The game of life is seen as a metric dynamical model $(((\{o, v\}^{\mathbb{Z} \times \mathbb{Z}}, d), \mathcal{V}), g)$. The points of the space are functions assigning the state of occupied or vacant to each point of $\mathbb{Z} \times \mathbb{Z}$. The dynamics g computes the update function of the game of life. The distance function is defined on $f, h \in \{o, v\}^{\mathbb{Z} \times \mathbb{Z}}$ as

$$d(f, h) = \inf \left\{ \frac{1}{k} \mid k \in \mathbb{N} \wedge \forall n, m < k. f(n, m) = h(n, m) \right\}$$

It is possible to show that g is continuous with respect to the topology induced by the metric. The valuation function may associate to proposition letters particular regions of interest, or patterns in the game grid.

Note 4.40. *There is probably some mistake in the formalisation of Example 4.39 given in [6]. It is unclear whether one should actually chose a different metric such as, starting from a 4×4 square around the origin, and then enlarging it. In particular, $k = 0$ is not in any sense "minimal" as the definition works also for negative values of k, but then one does not define a metric. It's strange that one starts with an already infinite agreement set (when $k = 0$, two functions must agree on all the points having negative coordinates!). Maybe there is something missing in the definition, e.g., $|n|, |m| < k$. This needs to be clarified.*

Note 4.41. *In Example 4.39, one could think to use a topo-model with points in $\mathbb{Z} \times \mathbb{Z}$ and proposition letters $\{o, v\}$. However, doing so, the dynamics of the system would need to change the valuation of propositions. In dynamic models and logics, as presented in [6], the propositions are fixed across time, and denote particular regions of interest, rather than properties changing over time. Thus it is important to use a function space to represent points.*

4.9.1 Dynamical logics

We now discuss how logics interact with dynamical models. We start from the topological case. Let $(((X, O), \mathcal{V}), g)$ be a dynamic topo-model. The logic has now a local flavour, since we are interested in discussing the orbits of specific points, and whether they will cross, or get trapped in, specific regions. Formally, this entails dealing only with spatial terms, not formulas.

Definition 4.42. The set of $\mathcal{DTL}$ terms is specified by the grammar:

$$\tau ::= p \mid \top \mid \bot \mid \neg\tau \mid \tau \wedge \tau \mid \tau \vee \tau \mid \Diamond\tau \mid \Box\tau \mid \bigcirc\tau \mid \Diamond_F\tau \mid \Box_F\tau$$

Terms are interpreted as sets of points.

Definition 4.43. The interpretation $\mathcal{E}^{\mathcal{M}}$ is defined by induction on terms. Cases for boolean operators, and propositions, are as usual; then we have:

$$\begin{array}{rclrcl} \mathcal{E}^{\mathcal{M}}(p) & = & \mathcal{V}(p) & \mathcal{E}^{\mathcal{M}} \bigcirc \tau & = & g^{-1}(\mathcal{E}^{\mathcal{M}}(\tau) \\ \mathcal{E}^{\mathcal{M}}(\Diamond_F\tau) & = & \bigcup_{i>0} g^{-i}(\mathcal{E}^{\mathcal{M}}(\tau)) & \mathcal{E}^{\mathcal{M}}(\Box_F\tau) & = & \bigcap_{i>0} g^{-i}(\mathcal{E}^{\mathcal{M}}(\tau)) \end{array}$$

The intuition behind the definition is that $\bigcirc\tau$ denotes the points that respect τ in the next step, that is, they belong to the inverse of the dynamics, applied to the set of points denoted by τ. Similarly, $\Diamond_F\tau$ and $\Box_F\tau$ denote points that *eventually*, or *always*, respectively, reach the set of points denoted by τ in a finite number of steps.

The case of metric dynamic logics is very similar. The only additional operator is $\exists^{\leq a}\tau$, with the intended metric interpretation, as in Section 4.4.

5 Discrete structures and closure spaces

Here we briefly introduce the mathematical theory of *closure spaces* and describe some preliminary results in the direction of spatial logics for discrete structures. A full study of the definable logics, their properties, and verification algorithms is left for future work. The material on closure spaces is mostly taken from [4], which enhances previous research presented in [8].

5.1 Closure spaces

Discrete spatial structures could be treated as in the continuous case, by defining a topology on top of the points of the structure. However, by doing so, one does not gain much advantage, as the closure operator, responsible for the meaning of the logical operator $\Diamond$, is idempotent in topological spaces. This assumption becomes too stringent for discrete structures. For example, in the case of regular grids, it is natural to interpret closure as the operation of enlarging a set of points by one step (in all possible directions) on the grid. Such interpretation is not idempotent. By removing the idempotency assumption, *closure spaces* are used in place of topological spaces.

Definition 5.1. A *closure space* is a pair $(X, \mathcal{C})$ where X is a set, and $\mathcal{C} : 2^X \to 2^X$ assigns to each subset of X its *closure*, such that, for all $A, B \subseteq X$:

1. $\mathcal{C}(\emptyset) = \emptyset$;
2. $A \subseteq \mathcal{C}(A)$;
3. $\mathcal{C}(A \cup B) = \mathcal{C}(A) \cup \mathcal{C}(B)$.

Definition 5.2. In a closure space $(X, \mathcal{C})$, given $A \subseteq X$, the *interior* $\mathcal{I}(A)$ of A is the set $\overline{\mathcal{C}(\overline{A})}$.

Lemma 5.3. In a closure space, we have $\mathcal{I}(A) \subseteq A$.

Definition 5.4. Set $A \subseteq X$ is a *neighbourhood* of $x \in X$ if and only if $x \in \mathcal{I}(A)$.

Definition 5.5. Set A is *closed* if $A = \mathcal{C}(A)$. It is *open* if $A = \mathcal{I}(A)$.

Lemma 5.6. In a closure space, A is open if and only if $\overline{A}$ is closed.

Lemma 5.7. In a closure space, closure and interior are monotone operators over the inclusion order, that is, $A \subseteq B \implies \mathcal{C}(A) \subseteq \mathcal{C}(B)$ and $\mathcal{I}(A) \subseteq \mathcal{I}(B)$.

Lemma 5.8. The open sets of a closure space are closed under finite intersections, and arbitrary unions.

The axioms defining a closure space are also part of the definition of a *Kuratowski closure space*, which is an alternative definition of a *topological space*. More precisely, the only missing axiom that makes a closure space Kuratowski is idempotence[1], that is $\mathcal{C}(\mathcal{C}(A) = \mathcal{C}(A)$.

Definition 5.9. A *topological space* is a closure space where the closure operator is idempotent, that is, for all $A \subseteq X$, $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$.

The correspondence between the Kuratowski definition (Definition 5.9) and the open sets definition (Definition 2.11) can be sketched as follows. To view a topological space defined in terms of open sets as a closure space, one defines $\mathcal{C}(A)$ as the smallest closed set containing A. For the converse, one uses the definition of an open set in a closure space, as given in Definition 5.5 (noting that closure is already assumed to be idempotent, by the Kuratowski definition).

5.2 Graphs as closure spaces

Discrete structures typically come in the form of a graph. A graph is described by its set of nodes X and its *connectedness* binary relation $R \subseteq X \times X$. A closure operator $\mathcal{C}_R$ can be derived from R as follows.

[1] When recovering the definition of a topological space via open sets from the Kuratowski definition, it is noteworthy that the preservation of binary unions is sufficient to prove that *arbitrary* unions of *open* sets are open.

Definition 5.10. Given a set X and a relation $R \subseteq X \times X$, define the closure operator

$$\mathcal{C}_R(A) = A \cup \{x \in X \mid \exists a \in A.(x, a) \in R\}$$

Proposition 5.11. The pair $(X, \mathcal{C}_R)$ is a closure space.

Closure operators obtained by Definition 5.10 are not necessarily idempotent. This is intimately related to reflexivity and transitivity of R, as shown by Lemma 11 in [4], that we rephrase below.

Lemma 5.12. The operator $\mathcal{C}_R$ is idempotent if and only if the reflexive closure $R^=$ of R is transitive.

Note that, when R is transitive, so is $R^=$, thus $\mathcal{C}_R$ is idempotent. The vice-versa is not true, as one may have $(x, y) \in R$, $(y, x) \in R$, but $(x, x) \notin R$.

5.3 Quasi-discrete structures

We now discuss interesting structures that do not necessarily have idempotent closure. See also Lemma 9 of [4] and the subsequent statements. We shall see that there is a very strong relation between the definition of a *quasi-discrete* space, given below, and graphs.

Definition 5.13. A closure space is *quasi-discrete* if and only if one of the following two equivalent conditions hold:

- each $x \in X$ has a *minimal neighbourhood*, that is, there is a neighbourhood $N_x \subseteq X$ of x (see Definition 5.4), which is included in all other neighbourhoods of x;
- for each $A \subseteq X$, $\mathcal{C}(A) = \bigcup_{a \in A} \mathcal{C}(\{a\})$.

The following is proved as Theorem 1 in [4].

Theorem 5.14. *A closure space $(X, \mathcal{C})$ is quasi-discrete if and only if there is a relation $R \subseteq X \times X$ such that $\mathcal{C} = \mathcal{C}_R$.*

Example 5.15. Existence of minimal neighbourhoods does not depend on finiteness of the space, and not even they depend on existence of a "closest element" for each point. To see this, consider the rational numbers $\mathbb{Q}$, equipped with the relation $\leq$. Such a relation is reflexive and transitive, thus the closure space $(\mathbb{Q}, \mathcal{C}_{\leq})$ is topological and quasi-discrete.

Example 5.16. Another example exhibiting minimal neighbourhoods in absence of closest elements is the following. Consider the rational numbers $\mathbb{Q}$ equipped with the relation $R = \{(x, y) \mid |x - y| \leq 1\}$, and let us look at the closure space $(\mathbb{Q}, \mathcal{C}_R)$. R is reflexive but not transitive, hence the obtained closure space is quasi-discrete but not topological. We have $\mathcal{C}_R(A) = \{x \in \mathbb{Q} \mid \exists a \in A.|a - x| \leq 1\}$. Consider a point x. For x to be included in $\mathcal{I}(A)$, a set A must include all the points whose distance from x is less or equal than 1, in other words, it must be true that $[x - 1, x + 1] \subseteq A$. To see this, suppose that there is $z \notin A$ such that $|z - x| \leq 1$. Then $x \in \mathcal{C}_R(\{z\})$, thus since $\overline{A} = \overline{A} \cup \{z\}$, we have $x \in \mathcal{C}_R(\overline{A})$, and therefore $x \notin \mathcal{I}(A) = \overline{\mathcal{C}_R(\overline{A})}$. The minimal neighbourhood is thus $N_x = [x - 1, x + 1]$. It is easily verified that $\mathcal{I}(N_x) = \{x\}$. In other words, each point x has a minimal neighbourhood N_x, and there is no other point y such that N_x is a neighbourhood of y. However, there are infinitely many points belonging to N_x.

Example 5.17. An example of a topological closure space which is not quasi-discrete is the set of real numbers equipped with the Euclidean topology (the topology induced by arbitrary union and finite intersection of open intervals). To see that the space is not quasi-discrete, one applies Definition 5.13. Consider an open interval (x, y). We have $\mathcal{C}((x, y)) = [x, y]$, but for each point z, we also have $\mathcal{C}(z) = [z, z] = \{z\}$. Therefore $\bigcup_{z \in (x,y)} \mathcal{C}(z) = \bigcup_{z \in (x,y)} \{z\} = (x, y) \neq [x, y]$.

Example 5.18. The reader may think that quasi-discreteness is also related to the space having a smaller cardinality than that of the real numbers. This is not the case. To see this, just equip the real numbers with an arbitrary relation, e.g., the relation $\leq$, in a similar way to Example 5.15. The obtained closure space is quasi-discrete.

Summing up, whenever one starts from an arbitrary relation $R \subseteq X \times X$, the obtained closure space $(X, \mathcal{C}_R)$ enjoys minimal neighbourhoods, and the closure of a set A is the union of the closure of the singletons composing A. Furthermore, such nice properties are only true in a closure space when there is some R such that the closure operator of the space is derived from R.

5.4 Boundaries

In [3], a discrete variant of the topological definition of the boundary of a set A is given, for the case where a closure operator is derived by Definition 5.10 from a reflexive and symmetric relation. Therein, in Lemma 5, it is proved that the definition coincides with the one we provide below. The latter is entierly defined in terms of closure and interior, and coincides with the definition of boundary in a topological space, so we prefer to adopt it for the general case of a closure space. Moreover, in discrete spaces (such as, grids) it sometimes makes sense to consider just the part of the boundary of a set A, which lies entirely within, or outside, A itself. We also define these notions.

Definition 5.19. In a closure space $(X, \mathcal{C})$, the *boundary* of $A \subseteq X$ is defined as $\mathcal{B}(A) = \mathcal{C}(A) \setminus \mathcal{I}(A)$. The *interior boundary* is $\mathcal{B}^-(A) = A \setminus \mathcal{I}(A)$, and the *closure boundary* is $\mathcal{B}^+(A) = \mathcal{C}(A) \setminus A$.

Proposition 5.20. The following equations hold in a closure space:

$$\mathcal{B}(A) = \mathcal{B}^+(A) \cup \mathcal{B}^-(A) \tag{1}$$
$$\mathcal{B}^+(A) \cap \mathcal{B}^-(A) = \emptyset \tag{2}$$
$$\mathcal{B}(A) = \mathcal{B}(\overline{A}) \tag{3}$$
$$\mathcal{B}^+(A) = \mathcal{B}^-(\overline{A}) \tag{4}$$
$$\mathcal{B}^+(A) = \mathcal{B}(A) \cap \overline{A} \tag{5}$$
$$\mathcal{B}^-(A) = \mathcal{B}(A) \cap A \tag{6}$$
$$\mathcal{B}(A) = \mathcal{C}(A) \cap \mathcal{C}(\overline{A}) \tag{7}$$

Proof. (of Proposition 5.20)

Equation 1: $\mathcal{B}(A) = \mathcal{C}(A) \setminus \mathcal{I}(A) = \quad [\mathcal{I}(A) \subseteq A, \forall A, B, C. B \subseteq C \implies A \setminus B = (A \setminus C) \cup (C \setminus B)] \quad (\mathcal{C}(A) \setminus A) \cup (A \setminus \mathcal{I}(A)) = \mathcal{B}^+(A) \cup \mathcal{B}^-(A)$

Equation 2: $\mathcal{B}^+(A) \cap \mathcal{B}^-(A) = (\mathcal{C}(A) \setminus A) \cap (A \setminus \mathcal{I}(A)) = \quad [\mathcal{C}(A) \setminus A \subseteq \overline{A}, A \setminus \mathcal{I}(A) \subseteq A] \quad \emptyset$

Equation 3: $\mathcal{B}(A) = \mathcal{C}(A) \setminus \mathcal{I}(A) = \overline{\mathcal{I}(\overline{A})} \setminus \overline{\mathcal{C}(\overline{A})} = \mathcal{C}(\overline{A}) \setminus \mathcal{I}(\overline{A}) = \mathcal{B}(\overline{A})$

Equation 4: $\mathcal{B}^-(\overline{A}) = \overline{A} \setminus \mathcal{I}(\overline{A}) = \overline{A} \setminus \overline{\mathcal{C}(A)} = \mathcal{C}(A) \setminus A = \mathcal{B}^+(A)$

Equation 5: $\mathcal{B}^+(A) = \mathcal{C}(A) \setminus A = \quad [\mathcal{I}(A) \subseteq A] \quad (\mathcal{C}(A) \setminus \mathcal{I}(A)) \setminus A = \mathcal{B}(A) \setminus A = \mathcal{B}(A) \cap \overline{A}$

Equation 6:
$\mathcal{B}^-(A) = \quad [\text{Equation 4}] \quad \mathcal{B}^+(\overline{A}) = \quad [\text{Equation 5}] \quad \mathcal{B}(\overline{A}) \cap A = \quad [\text{Equation 3}] \quad \mathcal{B}(A) \cap A$

Equation 7: $\mathcal{B}(A) = \mathcal{C}(A) \setminus \mathcal{I}(A) = \mathcal{C}(A) \cap \overline{\mathcal{I}(A)} = \mathcal{C}(A) \cap \mathcal{C}(\overline{A})$ □

6 Conclusions: towards discrete spatial logics and model checking

The literature study summarised in this report is meant to be applied to automated verification (e.g., *model checking*) of spatially distributed systems, especially in the context of mean-field/fluid-flow semantics of *collective adaptive systems*. For tractability reasons, discrete, finite models seem to be preferred to continuous models. Therefore, future developments ought to be grounded in the study of approaches to spatial logics for discrete structures. Current work in progress is focused on defining variants of spatio-temporal logics, interpreted in closure spaces, so that the $\Diamond$ modal operator is interpreted as closure. The descriptive possibilities of the obtained languages, extensions with bounded and unbounded *until* operators, and possibly global properties, are the aims of this research line. Once established these theoretical foundations, investigation will be directed to automated verification in the presence of (discrete) spatial and temporal information. Further extensions of interest include quantitative analysis, using metrics to characterise quantitiative properties of a spatially distributed system (e.g., one may study distances, probabilities, or costs in such a scenario), or characterising the probability of events through probabilistic logics, or statistical model checking.

References

[1] Marco Aiello, Ian Pratt-Hartmann, and Johan van Benthem, editors. *Handbook of Spatial Logics*. Springer, 2007.

[2] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic*. Cambridge University Press, New York, NY, USA, 2001.

[3] Antony Galton. The mereotopology of discrete space. In Christian Freksa and David M. Mark, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science*, volume 1661 of *Lecture Notes in Computer Science*, pages 251–266. Springer Berlin Heidelberg, 1999.

[4] Antony Galton. A generalized topological view of motion in discrete space. *Theoretical Computer Science*, 305(1-3):111–134, 2003.

[5] Peter T. Johnstone. *Sketches of an elephant : a topos theory compendium. Vol. 1.* Oxford Logic Guides. Clarendon Press, Oxford, 2002. Autre tirage : 2008.

[6] Roman Kontchakov, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev. Spatial logic + temporal logic = ? In Aiello et al. [1], pages 497–564.

[7] Philip Kremer and Grigori Mints. Dynamic topological logic. In Aiello et al. [1], pages 565–606.

[8] Michael B. Smyth and Julian Webster. Discrete spatial models. In Aiello et al. [1], pages 713–798.

[9] Johan van Benthem and Guram Bezhanishvili. Modal logics of space. In Aiello et al. [1], pages 217–298.