QUANTICOL

A Quantitative Approach to Management and Design of Collective and Adaptive Behaviours

http://www.quanticol.eu

TR-QC-2-2014

Differential Ordinary Lumpability in Markovian Process Algebra

Revision: 0.1; Feb 4, 2014

Author(s): Giulio Iacobelli, Mirco Tribastone, Andrea Vandin (SOTON)

Publication date: Feb 4, 2014

Funding Scheme: Small or medium scale focused research project (STREP)
Topic: ICT-2011 9.10: FET-Proactive 'Fundamentals of Collective Adaptive Systems' (FOCAS)
Project number: 600708

Coordinator: Jane Hillston (UEDIN)
e-mail: Jane.Hillston@ed.ac.uk
Fax: +44 131 651 1426

Part. no.	Participant organisation name	Acronym	Country
1 (Coord.)	University of Edinburgh	UEDIN	UK
2	Consiglio Nazionale delle Ricerche – Istituto di Scienza e Tecnologie della Informazione "A. Faedo"	CNR	Italy
3	Ludwig-Maximilians-Universität München	LMU	Germany
4	Ecole Polytechnique Fédérale de Lausanne	EPFL	Switzerland
5	IMT Lucca	IMT	Italy
6	University of Southampton	SOTON	UK

COOPERATION

Contents

Abstract

We present a theory of aggregation for ordinary differential equations (ODEs) induced by fluid semantics for Markovian process algebra. We introduce *differential ordinary lumpability* (DOL) as an equivalence relation which induces a partition over the ODEs of a model, whereby the sum of the ODEs belonging to the same partition block is equivalent to a single ODE. We study two variants of DOL, offering a tradeoff between the degree of coarsening that can be achieved and the preservation of compositional properties of the aggregation. Both variants are characterised in terms of two simple symmetries that can be verified using syntactic checks on the process term.

1 Introduction

In Markovian process algebra, fluid semantics interpret a term with a system of coupled ordinary differential equations (ODEs). This has proven useful in approximating the stochastic behaviour of models consisting of groups of many independent replicas of sequential components characterised by small local state spaces. The size of the underlying continuous-time Markov chain (CTMC) is well known to be at worst exponential in the population of such components. Instead, the fluid semantics defines a single ODE for each local state, independently from the multiplicities. The solution to each ODE estimates the time-course evolution of average population of components in that state (e.g., [10, 4, 12]).

In this paper we develop a theory of aggregation for ODE systems induced by a process algebra with fluid semantics. To illustrate and put in context our contribution, let us draw a parallel with established results of aggregation of CTMCs obtained from a Markovian semantics (e.g., [3, 9, 8]). This has involved finding suitable process algebraic behavioural relations that induce a partition of the CTMC state space which satisfies the property of *ordinary lumpability* [11, 2]: a smaller CTMC can be constructed where each state (a *macro-state*) is the representative of the states in a partition block; the probability of being in a macro-state is equal to the sum of the probabilities of being in the block's states.

Here we proceed in an analogous fashion. We introduce *differential ordinary lumpability* (DOL), an equivalence relation over the local states of a process algebra model that captures symmetries in the fluid semantics according to the well-known notion of *exact lumpability* for ODEs [16]. Specifically, given an ODE system and a partition of its ODEs, the system can be rewritten in terms of the variables aggregated according to the partition.

We present two versions of DOL. The first variant is more effective in terms of model reduction because discriminates less behaviour. For instance, it may be able to relate local states that are equal up to a renaming or a collapsing of some action types. While this can yield coarser aggregations, it clearly does not allow for compositional reasoning. For this reason, we also develop a stronger variant which is shown to be a congruence with respect to parallel composition. In both cases, establishing DOL involves *semantic checks*, i.e., tests of equalities of symbolic expressions depending on ODE variables. Here we characterise these two variants of DOL in terms of properties which only require syntactic checks on the process term, which has the potential to be more efficient to implement on a computer.

Synopsis. Section 2 introduces the necessary background material and our fluid calculus. Section 3 discusses DOL, and relates it to the exact lumpability of ODEs. Section 4 provides the characterisation of DOL, while Section 5 provides the congruent variant of DOL, as well as its characterisation. Section 6 discusses related work. Finally, Section 7 concludes the paper.[1]

2 Preliminaries

Our grammar has two levels. The first level specifies a *fluid atom*, i.e. a sequential process evolving over a discrete state space. Let $\mathcal{A}$ be the set of all actions and $\mathcal{K}$ the set of all fluid atoms. Each $P \in \mathcal{K}$ is a constant, $P \stackrel{def}{=} \sum_{i \in I} (\alpha_i, r_i).P_i$ where I is an index set, $\alpha_i \in \mathcal{A}$, $r_i > 0$ is a rate, and $P_i \in \mathcal{K}$. The standard transition relation can be defined, writing e.g., $P \xrightarrow{(\alpha_i, r_i)} P_i$. We define the *derivative set* of P, denoted by $ds(P)$ as the smallest set of atoms reached by P through such a relation. This is the fluid atom's local state space, and we may thus refer to $ds(P)$ as to the *local states* of the fluid atom P. Informally, a fluid atom is meant to represent an individual component in a group of many identical ones, i.e., a parallel composition $P \mid P \mid \cdots \mid P$ in a process algebra with a Markovian semantics. Furthermore, we define the *apparent rate* of P for action α as $r_\alpha(P) \triangleq \sum_{P \xrightarrow{(\alpha, r_i)} P'} r_i$; this is the total rate at which a local state can perform an action. For $P \in \mathcal{K}$ and $S \subseteq \mathcal{K}$, we define the the *total conditional transition rate* from P to S as $q[P, S, \alpha] \triangleq \sum_{P' \in S} \sum_{P \xrightarrow{(\alpha, r_i)} P'} r_i$.

We now define the second level of the grammar. We call this a Fluid Extended Process Algebra (FEPA) model because it improves the expressiveness of Fluid Process Algebra of [17]. It parameterises the parallel operator with a binary *synchronisation function*, denoted by $\mathcal{H}(\cdot, \cdot)$. We support two such functions, $\mathcal{H} = \min$ and $\mathcal{H} = \cdot$ (product). With the former we recover PEPA [9] (and [17]), while the latter can be used for the modelling of chemical reaction networks, and can be seen as the fluid counterpart of the process algebra in [3]. In this respect, fluid atoms correspond to, e.g., jobs and servers in a computing system, or molecular species in a chemical reaction network.

Definition 1 (FEPA model)**.** *A FEPA model $\mathcal{M}$ is given by the grammar*

$$\mathcal{M} ::= P \mid \mathcal{M} \parallel_L^{\mathcal{H}} \mathcal{M}\,, \qquad \text{with } L \subseteq \mathcal{A} \text{ and } P \in \mathcal{K}.$$

For any two fluid atoms P_1 and P_2 in $\mathcal{M}$, we require $ds(P_1) \cap ds(P_2) = \emptyset$. Requiring pairwise disjoint derivative sets is without loss of generality, see [17]. Since a fluid atom is a representative of a group of sequential components of the same type, the specification is completed by fixing the group size.

[1]The technical results given in the paper are proved in the enclosed appendix.

Definition 2 (Population functions)**.** *Let $\mathcal{M}$ be a FEPA model. We denote by $\mathcal{B}(\mathcal{M})$ the union of all the derivatives of the fluid atoms in a FEPA model. We define $\nu_0 : \mathcal{B}(\mathcal{M}) \to \mathbb{N}_0$, as the* initial population function *of $\mathcal{M}$. Furthermore, a* fluid population function *(or just* population function*) for $\mathcal{M}$ is defined as $\nu : \mathcal{B}(\mathcal{M}) \to \mathbb{R}_{\geq 0}$.*

Running example (step 1/10). Let us consider the FEPA model $\mathcal{M}_{RE} \triangleq P_1 \parallel^{\mathcal{H}}_{\{\alpha\}} Q_1$, where

$$P_1 \overset{def}{=} (\beta, r).P_2 + (\beta, r).P_3 \qquad P_2 \overset{def}{=} (\gamma, s).P_1 + (\alpha, l).P_3 \qquad P_3 \overset{def}{=} (\xi, s).P_1$$
$$Q_1 \overset{def}{=} (\beta, 2r).Q_2 \qquad Q_2 \overset{def}{=} (\eta, s).Q_1 + (\alpha, m).Q_2$$

Thus we have $ds(P_1) = \{P_1, P_2, P_3\}$, $ds(Q_1) = \{Q_1, Q_2\}$, and $\mathcal{B}(\mathcal{M}_{RE}) = \{P_1, P_2, P_3, Q_1, Q_2\}$. Let us fix $\nu_0(P_1) = N_P$, $\nu_0(Q_1) = N_Q$, and $\nu_0(P_2) = \nu_0(P_3) = \nu_0(Q_2) = 0$, for some integers N_P and N_Q. Intuitively, this model represents a discrete-state stochastic model with an initial state with N_P copies of P_1 in parallel over empty cooperation sets, where each may synchronise (over action α) with one of the N_Q copies of Q_1 in parallel. The fluid approximation estimates the expected population of sequential components at each time point that can be found in each of the local states of $\mathcal{B}(\mathcal{M}_{RE})$. □

We are now ready to provide the semantics for interaction in FEPA.

Definition 3 (Population-dependent apparent rate)**.** *Let $\mathcal{M}$ be a FEPA model, ν be a population function, $L \subseteq \mathcal{A}$ a set of actions, and $\alpha \in \mathcal{A}$ an action. The apparent rate of α in $\mathcal{M}$ with respect to ν is recursively defined as*

$$r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_{L} \mathcal{M}_2, \nu) \triangleq \begin{cases} \mathcal{H}\big(r_\alpha(\mathcal{M}_1, \nu), r_\alpha(\mathcal{M}_2, \nu)\big), & \text{if } \alpha \in L, \\ r_\alpha(\mathcal{M}_1, \nu) + r_\alpha(\mathcal{M}_2, \nu) & \text{if } \alpha \notin L, \end{cases}$$
$$r_\alpha(P, \nu) \triangleq \sum_{P' \in \mathcal{B}(P)} \nu_{P'} r_\alpha(P') \; .$$

Running example (step 2/10). In $\mathcal{M}_{RE}$ we have $r_\beta(P_1, \nu) = 2r\,\nu_{P_1}$ and $r_\alpha(P_1, \nu) = l\,\nu_{P_2}$. The interpretation is that the population-dependent apparent rate of a fluid atom gives the total rate at which an action can be performed by all its local states. Let us remark that it is a *symbolic expression*, depending on a population function ν. The rate is affected by the presence of a synchronisation, e.g., $r_\alpha(\mathcal{M}_{RE}, \nu) = \min(l\,\nu_{P_2}, m\,\nu_{Q_2})$ or $r_\alpha(\mathcal{M}_{RE}, \nu) = l\,\nu_{P_2} \cdot m\,\nu_{Q_2}$, depending on the chosen synchronisation function. This is intended as the overall speed at which action α is performed in the model; e.g., it is zero if either ν_{P_2} or ν_{Q_2} is zero, capturing the blocking effect of synchronisation for both choices of $\mathcal{H}$. □

Using the notion of apparent rates, the next definition identifies the class of well-posed models, taken from [17].

Definition 4 (Well-posedness)**.** *A FEPA model $\mathcal{M}$ is* well-posed *if for all occurrences $\mathcal{M}_1 \parallel^{\mathcal{H}}_{L} \mathcal{M}_2$ in $\mathcal{M}$, and for all $\alpha \in L$ it holds that $\exists \nu_1 : r_\alpha(\mathcal{M}_1, \nu_1) > 0$, and $\exists \nu_2 : r_\alpha(\mathcal{M}_2, \nu_2) > 0$. A model that is not well posed is said to be* ill-posed*.*

In essence, an ill-posed model defines a composition where an action type α is declared as synchronised, but it cannot be performed by at least one of the two operands, because the apparent rate is zero. In the remainder we assume to always work with well-posed FEPA models. However this is without loss of generality because, similarly to [18], any ill-posed FEPA model can be shown to be syntactically transformed into a well-posed one yielding an equivalent underlying ODE system.

The next definition provides the rate at which the population of a specific local state performs an action.

Definition 5 (Population-dependent component rate)**.** *Let $\mathcal{M}$ be a FEPA model, ν be a population function, $\alpha \in \mathcal{A}$ an action, and $P \in \mathcal{B}(\mathcal{M})$. The component rate of P due to α-type activities within the model $\mathcal{M}$ with respect to ν is recursively defined as*

$$\mathcal{R}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu, P) \triangleq \begin{cases} \mathcal{R}_\alpha(\mathcal{M}_i, \nu, P) \dfrac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_i, \nu)} , & \text{if } P \in \mathcal{B}(\mathcal{M}_i) \text{ and } \alpha \in L, \text{ for } i = 1, 2 , \\ \mathcal{R}_\alpha(\mathcal{M}_i, \nu, P) , & \text{if } P \in \mathcal{B}(\mathcal{M}_i) \text{ and } \alpha \notin L, \text{ for } i = 1, 2 , \end{cases}$$

$$\mathcal{R}_\alpha(P, \nu, P') \triangleq \begin{cases} \nu_{P'} r_\alpha(P') & \text{if } P' \in \mathcal{B}(P), \\ 0 & \text{otherwise} . \end{cases}$$

The terms of the form $\dfrac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_i, \nu)}$ *are defined as* 0 *when* $r_\alpha(\mathcal{M}_i, \nu) = 0$.

These are the constituent elements of the ODE system underlying a FEPA model according to the fluid semantics, defined next. For this definition and throughout the remainder of the paper we denote the derivative of ν using Newton's dot notation, namely, $\dot{\nu}$. To enhance readability, time t will be suppressed in the representation of ODEs, i.e., $\dot{\nu}$ denotes $\dot{\nu}(t)$ and ν denotes $\nu(t)$.

Definition 6 (Fluid semantics)**.** *Let $\mathcal{M}$ be a FEPA model. Let $E \subseteq \mathbb{R}^{\mathcal{B}(\mathcal{M})}$ and $f : E \to \mathbb{R}^{\mathcal{B}(\mathcal{M})}$ be the vector field whose components are defined as:*

$$f_P(\nu) \triangleq \sum_{\alpha \in \mathcal{A}} \left(\sum_{P' \in \mathcal{B}(\mathcal{M})} p_\alpha(P', P) \mathcal{R}_\alpha(\mathcal{M}, \nu, P') - \mathcal{R}_\alpha(\mathcal{M}, \nu, P) \right) , \tag{1}$$

for every $P \in \mathcal{B}(\mathcal{M})$, with $p_\alpha(P, P') \triangleq (1/r_\alpha(P)) \sum_{P \xrightarrow{(\alpha,r)} P'} r$.

The evolution of the population function $\nu(t)$ over time is governed by the ODE system

$$\dot{\nu} = f(\nu) \qquad \text{with initial condition (i.c.)} \qquad \nu(0) = \nu_0 . \tag{2}$$

We shall usually refer to (2) as the ODE system of $\mathcal{M}$ and to the vector field f in (1) as the vector field of $\mathcal{M}$.

Running example (step 3/10). For instance, in $\mathcal{M}_{RE}$ we have $R_\gamma(\mathcal{M}_{RE}, \nu, P_2) = s\,\nu_{P_2}$, and $R_\alpha(\mathcal{M}_{RE}, \nu, P_2) = \mathcal{H}(l\nu_{P_2}, m\nu_{Q_2})$. The ODE system of $\mathcal{M}_{RE}$ is

$$\begin{aligned} \dot{\nu}_{P_1} &= -2r\,\nu_{P_1} + s\,\nu_{P_2} + s\,\nu_{P_3} & \dot{\nu}_{P_2} &= -s\,\nu_{P_2} - \mathcal{H}(l\,\nu_{P_2}, m\,\nu_{Q_2}) + r\,\nu_{P_1} \\ \dot{\nu}_{Q_1} &= -2r\,\nu_{Q_1} + s\,\nu_{Q_2} & \dot{\nu}_{P_3} &= -s\,\nu_{P_3} + \mathcal{H}(l\,\nu_{P_2}, m\,\nu_{Q_2}) + r\,\nu_{P_1} \\ & & \dot{\nu}_{Q_2} &= -s\,\nu_{Q_2} + 2r\,\nu_{Q_1} \end{aligned} \tag{3}$$

By Definition 6, $\dot{\nu}_{Q_2}$ also has a contribution due to $R_\alpha(\mathcal{M}_{RE}, \nu, Q_2) = m\,\nu_{Q_2}$; this however cancels out due to the fact that (α, m) is a self-transition for Q_2. In general, a self-transition does not bring any contribution to the ODE of the related state. □

3 Differential Ordinary Lumpability

We are now ready to define DOL. Before discussing the details of the aggregation technique, we first provide an intuition using our running example.

Running example (step 4/10). Let us consider the partition $\mathcal{P}_{RE}$ of $\mathcal{B}(\mathcal{M}_{RE})$ defined as $\mathcal{P}_{RE} = \{S_1, S_2\}$, with $S_1 = \{P_1, Q_1\}$ and $S_2 = \{P_2, P_3, Q_2\}$. Summing the ODEs within each partition block we obtain

$$(\dot{\nu_{P_1} + \nu_{Q_1}}) = \dot{\nu}_{P_1} + \dot{\nu}_{Q_1} = -2r\,(\nu_{P_1} + \nu_{Q_1}) + s\,(\nu_{P_2} + \nu_{P_3} + \nu_{Q_2})$$
$$(\dot{\nu_{P_2} + \nu_{P_3} + \nu_{Q_2}}) = -s\,(\nu_{P_2} + \nu_{P_3} + \nu_{Q_2}) + 2r\,(\nu_{P_1} + \nu_{Q_1})$$

In this way we have found an ODE system of two equations that only depends on sums of variables of the original one. Thus, using the change of variable $\hat{\nu}_{S_1} = \nu_{P_1} + \nu_{Q_1}$ and $\hat{\nu}_{S_2} = \nu_{P_2} + \nu_{P_3} + \nu_{Q_2}$, a solution of (3) is also a solution of the aggregated ODEs:

$$\dot{\hat{\nu}}_{S_1} = -2r\hat{\nu}_{S_1} + s\hat{\nu}_{S_2} \qquad \dot{\hat{\nu}}_{S_2} = 2r\hat{\nu}_{S_1} - s\hat{\nu}_{S_2} \tag{4}$$

with i.c. $\hat{\nu}_{S_1}(0) = \nu_0(P_1) + \nu_0(Q_1)$, $\hat{\nu}_{S_2}(0) = \nu_0(P_2) + \nu_0(Q_3) + \nu_0(Q_2)$. □

Our approach builds on a general theory of aggregation of dynamical systems [16], which we briefly overview and recast to our notation. Let $\mathcal{M}$ be a FEPA model, let $\hat{n} \leq |\mathcal{B}(\mathcal{M})|$ be an integer and M be a $\hat{n} \times |\mathcal{B}(\mathcal{M})|$ real constant matrix with rank $\hat{n}$. Let us now consider some function $\hat{f} : E \to \mathbb{R}^{\hat{n}}$, where $E \subseteq \mathbb{R}^{\hat{n}}$. The *lumpability condition* $\hat{f} \circ M = M \circ f$, where $\circ$ is the operator function composition, is necessary and sufficient to hold that $\hat{v}(t)$, the solution to the aggregated ODE system $\dot{\hat{v}} = \hat{f}(\hat{v})$ (of size $\hat{n}$), satisfies $\hat{v}(t) = Mv(t)$. For this, it is necessary that $\hat{f} = M \circ f \circ \overline{M}$ holds, where $\overline{M} \in \mathbb{R}^{n \times \hat{n}}$ denotes the generalised right inverse of M satisfying $M\overline{M} = I_{\hat{n}}$, with $I_{\hat{n}} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ the identity.

The remainder of this section is devoted to defining the process-algebraic conditions to yield a partition of $\mathcal{B}(\mathcal{M})$ inducing an *aggregation matrix* M that satisfies the lumpability condition for such choice of $\hat{f}$.

Let us start by observing an analogy with *step 3/10* when a partition of the local states spaces is considered. The α-contributions $\mathcal{R}_\alpha(\mathcal{M}_{RE}, \nu, P_2) = \mathcal{R}_\alpha(\mathcal{M}_{RE}, \nu, P_3) = \mathcal{H}\,(l\,\nu_{P_2}, m\,\nu_{Q_2})$ only affect the local states P_2 and P_3, which are in the same partition block. These cancel out when writing the aggregated ODE system (4). Thus, actions with transitions that are only internal to a partition block can intuitively be seen as self-transitions at the block-level. These actions are characterised by the following definition.

Definition 7 ($\mathcal{P}$-External actions). *Let $\mathcal{M}$ be a FEPA model, let $\mathcal{P}$ be a partition of $\mathcal{B}(\mathcal{M})$. We define the set of $\mathcal{P}$-external actions as*

$$\mathcal{A}^{\mathcal{P}}_{ext} = \{\alpha \in \mathcal{A} \mid \exists S, S' \in \mathcal{P} : S \neq S' \quad and \quad \exists P \in S : q[P, S', \alpha] > 0\} \,.$$

Furthermore we define the set of $\mathcal{P}$-internal actions as $\mathcal{A}^{\mathcal{P}}_{int} = \mathcal{A} \setminus \mathcal{A}^{\mathcal{P}}_{ext}$.

Now, we consider the simple observation that, similarly to the Markovian semantics, ODEs are to a large extent agnostic to action types. Here we just need to distinguish between action types that are used to denote independent actions and those that are used for synchronisation.

Definition 8 (Dependent action set). *Let $\mathcal{M}$ be a FEPA model, let $P \in \mathcal{B}(\mathcal{M})$. The* dependent action set *of P in $\mathcal{M}$ is recursively defined as*

$$\mathcal{D}(P, \mathcal{M}) \triangleq \begin{cases} L \cup \mathcal{D}(P, \mathcal{M}_i)\,, & \text{if } \mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \text{ and } P \in \mathcal{B}(\mathcal{M}_i)\,, \\ \emptyset\,, & \text{if } \mathcal{M} = P \text{ or } P \notin \mathcal{B}(\mathcal{M})\,. \end{cases}$$

We say that α is independent *for a local state P in $\mathcal{M}$, if $\alpha \notin \mathcal{D}(P, \mathcal{M})$. Moreover, we define the* current dependent action set *of P in $\mathcal{M}$ as $\mathcal{CD}(P, \mathcal{M}) \triangleq \mathcal{D}(P, \mathcal{M}) \cap \mathcal{A}(P)$. Finally, for $\hat{\mathcal{A}} \subseteq \mathcal{A}$, the $\hat{\mathcal{A}}$-*restricted current dependent action set *is defined as $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) \triangleq \mathcal{CD}(P, \mathcal{M}) \cap \hat{\mathcal{A}}$.*

Abusing notation, for any $K \subseteq \mathcal{B}(\mathcal{M})$, we denote $\mathcal{D}(K,\mathcal{M}) \triangleq \bigcup_{P\in K} \mathcal{D}(P,\mathcal{M})$, $\mathcal{CD}(K,\mathcal{M}) \triangleq \bigcup_{P\in K} \mathcal{CD}(P,\mathcal{M})$, and $\mathcal{CD}^{\hat{\mathcal{A}}}(K,\mathcal{M}) \triangleq \bigcup_{P\in K} \mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M})$.

Intuitively, $\mathcal{D}(P,\mathcal{M})$ consists of the actions which P *may* use for synchronisation; $\mathcal{CD}(P,\mathcal{M})$ considers those actions through which P *actually* interacts. We wish to exploit this definition to aggregate local states that exhibit distinct independent actions, which however behave similarly within the model under consideration. For instance, let us consider $R_1 \stackrel{def}{=} (\alpha,r).R_2$, $R_2 \stackrel{def}{=} (\beta,s).R_1$, $G_1 \stackrel{def}{=} (\gamma,r).G_2$, $G_2 \stackrel{def}{=} (\delta,s).G_1$, and the FEPA process $R_1 \parallel^{\mathcal{H}}_{\emptyset} G_1$. It is easy to see that R_1 and G_1, and R_2 and G_2 have the same ODE even if they perform different actions, because all the action types are independent for any local state.

Besides playing a crucial role in the definition of ordinary fluid lumpability, dependent action sets identify the class of $\mathcal{A}$-*coherent* FEPA models.

Definition 9 ($\mathcal{A}$-coherence)**.** *A FEPA model $\mathcal{M}$ is $\mathcal{A}$-coherent iff:*

$$\forall \alpha \in \mathcal{D}(\mathcal{B}(\mathcal{M}),\mathcal{M}), \forall P \in \mathcal{B}(\mathcal{M}): \quad \alpha \notin \mathcal{D}(P,\mathcal{M}) \;\Rightarrow\; r_\alpha(P) = 0 \; .$$

Essentially, this allows us to rule out models having (at least) an action with the double role of appearing in the dependent action set of some local state and of being an independent action for some other local state. In the rest of this paper, we shall focus on the class of $\mathcal{A}$-coherent FEPA models. Nevertheless, this is not restrictive, as it is possible to transform any non-$\mathcal{A}$-coherent model into an $\mathcal{A}$-coherent one by resorting to a simple syntactic transformation reminiscent of the well-known α-renaming, as exemplified in the following.

Running example (step 5/10). Consider the FEPA model $\mathcal{M}_U \stackrel{def}{=} \mathcal{M}_{RE} \parallel^{\mathcal{H}}_{\emptyset} U_1$, where U_1 is defined as $U_1 \stackrel{def}{=} (\alpha,l).U_2$, $U_2 \stackrel{def}{=} (\gamma,s).U_1$. Note that $\mathcal{M}_U$ is not $\mathcal{A}$-coherent, as α is a dependent action in $\mathcal{M}_{RE}$, that is $\alpha \in \mathcal{D}(\mathcal{B}(\mathcal{M}_{RE}),\mathcal{M}_U)$; however α is independent for U_1, since $\alpha \notin \mathcal{D}(U_1,\mathcal{M}_U)$, and $r_\alpha(U_1) = l > 0$. However, it is easy to see that we can rename the independent α action of U_1 with a new label (e.g., α'), resulting in a model with the same ODEs. □

Definition 10 (Model influence)**.** *Let $\mathcal{M}$ be a FEPA model, ν be a population function on $\mathcal{B}(\mathcal{M})$, $\alpha \in \mathcal{A}$ and $P \in \mathcal{B}(\mathcal{M})$. The* model influence *upon P due to α-type activities within the model $\mathcal{M}$ with respect to ν is recursively defined as*

$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu, P) \triangleq \begin{cases} \mathcal{F}_\alpha(\mathcal{M}_i,\nu,P)\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2,\nu)}{r_\alpha(\mathcal{M}_i,\nu)} \; , & \text{if } P \in \mathcal{B}(\mathcal{M}_i) \text{ and } \alpha \in L, \text{ for } i = 1,2 \; , \\ \mathcal{F}_\alpha(\mathcal{M}_i,\nu,P) \; , & \text{if } P \in \mathcal{B}(\mathcal{M}_i) \text{ and } \alpha \notin L, \text{ for } i = 1,2 \; , \end{cases}$$

$$\mathcal{F}_\alpha(P,\nu,P') \triangleq \begin{cases} 1 & \text{if } P' \in \mathcal{B}(P) \; , \\ 0 & \text{otherwise} \; . \end{cases}$$

The terms of the form $\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2,\nu)}{r_\alpha(\mathcal{M}_i,\nu)}$ are defined as 0 *when $r_\alpha(\mathcal{M}_i,\nu) = 0$.*

This notion captures the effect that the environment exerts on the rate at which an action is performed by a process. For instance, it easily follows that, for any FEPA model $\mathcal{M}$, component $P \in \mathcal{B}(\mathcal{M})$, and action $\alpha \in \mathcal{A}$, it holds that $\mathcal{R}_\alpha(\mathcal{M},\nu,P) = \nu_P \cdot r_\alpha(P) \cdot \mathcal{F}_\alpha(\mathcal{M},\nu,P)$ for all $\nu \in \mathbb{R}^{\mathcal{B}(\mathcal{M})}$. In other words, the actual α-component rate for P is given by the rate at which P would evolve if it was independent, i.e., $\nu_P \cdot r_\alpha(P)$, weighted by the influence of the context on it, i.e., $\mathcal{F}_\alpha(\mathcal{M},\nu,P)$. Indeed, the following characterises this observation: An action is independent for a local state P iff the model does not influence its behaviour.

Proposition 1. *Let $\mathcal{M}$ be a FEPA model, $P \in \mathcal{B}(\mathcal{M})$ and $\alpha \in \mathcal{A}$. Then, $\alpha \notin \mathcal{D}(P,\mathcal{M}) \Leftrightarrow \mathcal{F}_\alpha(\mathcal{M},\nu,P) = 1$, for any ν.*

We are now ready to define differential ordinary lumpability (DOL).

Definition 11 (Differential ordinary lumpability)**.** *Let $\mathcal{M}$ be a well-posed FEPA model. Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{M}) \times \mathcal{B}(\mathcal{M})$ be an equivalence relation over the local states of $\mathcal{B}(\mathcal{M})$, and $\mathcal{P}$ be the partition of $\mathcal{B}(\mathcal{M})$ induced by $\mathcal{R}$. We say that $\mathcal{R}$ is a* differential ordinary lumpability *iff, for all $S \in \mathcal{P}$, whenever $P, Q \in S$, the three following conditions hold:*

(i) $\mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M}) \triangleq \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{ext}}(P,\mathcal{M}) = \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{ext}}(Q,\mathcal{M})$,

(ii) for all $\tilde{S} \in \mathcal{P}$, $\alpha \in \mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M})$, and ν,

$$q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P) = q[Q,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,Q) \ ,$$

(iii) for all $\tilde{S} \neq S \in \mathcal{P}$,

$$\sum_{\alpha \in \mathcal{A} \setminus \mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M})} q[P,\tilde{S},\alpha] = \sum_{\alpha \in \mathcal{A} \setminus \mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M})} q[Q,\tilde{S},\alpha] \ .$$

Furthermore, we refer to $\mathcal{P}$ as a differential ordinary lumpable partition *(DOLP).*

Condition (i) requires that local states within the same block interact with the rest of the model through the same $\mathcal{P}$-external dependent actions. It does not pose constraints on $\mathcal{P}$-internal actions, to capture the observation that their contributions cancel out when writing the aggregated ODE system. Also condition (ii) focuses on external current dependent actions. For *each* of them, it requires the equality between the rates with which the local states perform transitions towards any partition block. The rates are given as products of the total conditional transition rates and the model influences. Condition (iii) instead, considers independent actions; here the model influence does not show because it is always equal to one, by Proposition 1.

Using our running example, let us illustrate two properties of this definition.

Running example (step 6/10). In contrast to (ii), condition (iii) does not require the *per-action* equality of the rates, but only of the sum across all independent actions. For instance, the partition $\mathcal{P}_{RE}$ defined in *step 4/10* satisfies condition (iii) even if each of the local states of its block S_2, that is P_2, P_3 and Q_2, perform one and only distinct independent action, i.e. γ, ξ and η, respectively. Indeed, noticing that $\mathcal{A} \setminus \mathcal{CD}^{\mathrm{ext}}(S_2,\mathcal{M}) = \{\gamma,\xi,\eta\}$, it holds that $q[P_2,S_1,\gamma] = q[P_3,S_1,\xi] = q[Q_2,S_1,\eta] = s$. That is, we are able to aggregate the equations for P_2, P_3 and Q_2 even if they perform independent actions with different action types, because they have the same overall rate towards block S_1. □

Running example (step 7/10). Condition (iii) does not consider internal transitions. This is because internal transitions labelled with independent actions necessarily affect only local states within the same partition block, thus their contribution cancels out in the aggregated ODE system. This does not hold, in general, for internal transitions involving dependent actions. To see this, let us assume to weaken (ii) by requiring it only for all partitions $\tilde{S}$ *except* S, in order to disregard internal transitions. Let us consider the model $\mathcal{M}_{E'} = P_1 \parallel^{\mathcal{H}}_{\{\alpha\}} U_1$, where P_1 and U_1 are as in our running example, i.e. $P_1 \stackrel{def}{=} (\beta,r).P_2 + (\beta,r).P_3$, $P_2 \stackrel{def}{=} (\gamma,s).P_1 + (\alpha,l).P_3$ and $P_3 \stackrel{def}{=} (\xi,s).P_1$, and $U_1 \stackrel{def}{=} (\alpha,l).U_2$, and $U_2 \stackrel{def}{=} (\gamma,s).U_1$, and $\mathcal{H} = \min$. Then, the partition $\mathcal{P}' = \{\{P_1\},\{P_2,P_3\},\{U_1\},\{U_2\}\}$ can be shown to satisfy this weakened version of DOL, but it does not lead to an aggregated ODE system. This is because the fluid semantics yields the ODEs

$$\dot{\nu}_{U_1} = -\min\left(l\,\nu_{P_2}, l\,\nu_{U_1}\right) + s\,\nu_{U_2} \qquad\qquad \dot{\nu}_{U_2} = -s\,\nu_{U_2} + \min\left(l\,\nu_{P_2}, l\,\nu_{U_1}\right)$$

but it is not possible to express them in terms of the aggregation $\nu_{P_2} + \nu_{P_3}$. □

Finally, let us observe that DOL captures other somewhat degenerate forms of ODE aggregation. In particular, we recover a principle of *conservation of mass*: Summing the ODEs of the local states of a fluid atom P always yields the aggregated ODE $\sum_{P' \in \mathcal{B}(P)} \dot{\nu}_{P'} = 0$. This is due to the disjointness assumption of the fluid atoms, which implies that there exist no transitions among local states of different fluid atoms, thus the sum of the population of the local states of a fluid atom is constant.

Running example (step 8/10). Consider the model $\mathcal{M}_{E'}$ in *step 7/10*, and the partition $\mathcal{P}_{\text{deg}} = \{\mathcal{B}(P_1), \mathcal{B}(U_1)\}$ of $\mathcal{B}(\mathcal{M}_{E'})$. Summing the ODEs of the local states in $\mathcal{B}(U_1)$, we obtain

$$\dot{\nu}_{U_1} + \dot{\nu}_{U_2} = -\min\left(l\,\nu_{P_2}, l\,\nu_{U_1}\right) + s\,\nu_{U_2} - s\,\nu_{U_2} + \min\left(l\,\nu_{P_2}, l\,\nu_{U_1}\right) = 0\ ,$$

similarly for the ODEs of the local states in $\mathcal{B}(P_1)$. □

Another degenerate ODE aggregation consists in choosing, for any model $\mathcal{M}$, the trivial partition $\mathcal{P} = \{\mathcal{B}(\mathcal{M})\}$. This is always a DOLP because $\mathcal{A}^{\mathcal{P}}_{\text{ext}} = \emptyset$, which makes conditions (i)–(iii) trivially satisfied. The aggregated ODE system is $\sum_{P \in \mathcal{B}(\mathcal{M})} \dot{\nu}_P = 0$, which simply says that the total population within the model is constant.

Finally, the following theorem gives our desired result whereby DOL implies ODE aggregation.

Theorem 1. *Let $\mathcal{M}$ ba a well-posed and $\mathcal{A}$-coherent FEPA model, $\mathcal{P}$ be a DOLP of $\mathcal{B}(\mathcal{M})$. Let $M_{\mathcal{P}}$ be the aggregation matrix induced by $\mathcal{P}$ on $\mathcal{M}$, that is the $|\mathcal{P}| \times |\mathcal{B}(\mathcal{M})|$ matrix with entries 0 or 1 defined as*

$$(M_{\mathcal{P}})_{i,j} \triangleq \begin{cases} 1 & \text{if } P_j \in S_i\ , \\ 0 & \text{otherwise}\ , \end{cases}$$

where S_i, with $i \in \{1, \ldots, |\mathcal{P}|\}$, is a block of the partition $\mathcal{P}$ and P_j, with $j \in \{1, \ldots, |\mathcal{B}(\mathcal{M})|\}$, is a local state of the model $\mathcal{M}$. Let $\hat{f} \triangleq M_{\mathcal{P}} \circ f \circ \overline{M}_{\mathcal{P}}$. Then $\hat{\nu}(t)$, solution of the ODE system

$$\dot{\hat{\nu}} = \hat{f}(\hat{\nu}), \qquad \text{with initial condition} \quad \hat{\nu}(0) = M_{\mathcal{P}}\nu_0\ ,$$

satisfies $\hat{\nu}(t) = M_{\mathcal{P}}\nu(t)$, where $\nu(t)$ is solution of the ODE system of $\mathcal{M}$.

4 Characterisation of Differential Ordinary Lumpability

We study a characterisation of DOL in terms of two kinds of symmetries among the local states of a FEPA model. One symmetry is *local*, and regards transition rates of the local states; the other, instead, is *global*, and captures structural properties concerning the possible interactions between a local state and the environment. In addition of being of interest per se, this characterisation also provides an alternative route to verifying DOL, which can be more convenient in an implementation. Indeed, Definition 11 requires *ODE semantic checks*, in the sense that it involves, for any possible population function, the computation of all model influences, which are the components of the vector field of the underlying ODEs. This requires the symbolic evaluation of such influences in condition (ii). By contrast, our characterisation only requires *syntactic checks* that do not involve population functions, and only consider the parsing of syntax tree of a model and the computation of conditional transition rates.

We start with capturing local symmetry with the following.

Definition 12 ($\mathcal{CD}$-strong equivalence)**.** *Let $\mathcal{M}$ be a well-posed FEPA model. Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{M}) \times \mathcal{B}(\mathcal{M})$ be an equivalence relation over the local states of $\mathcal{B}(\mathcal{M})$, and $\mathcal{P}$ be the partition of $\mathcal{B}(\mathcal{M})$ induced by $\mathcal{R}$. We say that $\mathcal{R}$ is a $\mathcal{CD}$–*strong equivalence *iff for all $S \in \mathcal{P}$, whenever $P, Q \in S$, the following conditions hold:*

(i) $\mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M}) \triangleq \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\mathrm{ext}}}(P,\mathcal{M}) = \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\mathrm{ext}}}(Q,\mathcal{M})$,

(ii) for all $\tilde{S} \in \mathcal{P}$, *and for all* $\alpha \in \mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M})$

$$q[P,\tilde{S},\alpha] = q[Q,\tilde{S},\alpha]\ ,$$

(iii) for all $\tilde{S} \neq S \in \mathcal{P}$,

$$\sum_{\alpha\in\mathcal{A}\setminus\mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M})} q[P,\tilde{S},\alpha] = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{CD}^{\mathrm{ext}}(S,\mathcal{M})} q[Q,\tilde{S},\alpha]\ .$$

Furthermore we say that P *and* Q *are* $\mathcal{CD}$-strong equivalent *if there exists a* $\mathcal{CD}$*-strong equivalence relating them.*

We call it $\mathcal{CD}$-strong equivalence because it reduces to a definition in the style of Larsen and Skou [13] that resembles PEPA's *strong equivalence* [9]. Interestingly, strong equivalence has been shown to be sufficient to yield ordinary lumpability at the level of the underlying CTMC [9]. Here, instead, $\mathcal{CD}$-strong equivalence has to be paired with another symmetry in order to be a sufficient condition for the lumpability at the level of the underlying ODEs. Let us notice, in fact, that it is similar to DOL, except that it disregards the model influence. This is encoded in the notion of $\mathcal{CD}$-context, thus formalising the global symmetry existing among the local states of a block of a DOLP.

Definition 13 ($\mathcal{CD}$-context)**.** *Let* $\mathcal{M}$ *be a well-posed FEPA model, and* $\hat{\mathcal{A}}$ *be a set of actions. Let* $P,Q \in \mathcal{B}(\mathcal{M})$*. We say that* P*,* Q *are in* $\mathcal{CD}$–context *with respect to* $\hat{\mathcal{A}}$ *iff* $\mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M})$ *and one of the two following conditions hold:*

(i) it does not exist any occurrence $\overline{\mathcal{M}} = \mathcal{M}_1 \parallel^{\mathcal{H}}_{L} \mathcal{M}_2$ *within* $\mathcal{M}$ *with* $P \in \mathcal{B}(\mathcal{M}_1)$*, and* $Q \in \mathcal{B}(\mathcal{M}_2)$ *(or vice versa), or*

(ii) if such occurrence exists, then $\mathcal{CD}^{\hat{\mathcal{A}}}(P,\overline{\mathcal{M}}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\overline{\mathcal{M}}) = \emptyset$*.*

With the notion of $\mathcal{CD}$-context, $\mathcal{CD}$-strong equivalence characterises DOLP.

Theorem 2. *Let* $\mathcal{M}$ *be a well-posed FEPA model and* $\mathcal{P}$ *a partition of* $\mathcal{B}(\mathcal{M})$*.* $\mathcal{P}$ *is differential ordinary lumpable if and only if there exists a* $\mathcal{CD}$*–strong equivalence inducing the partition* $\mathcal{P}$*, and the local states of each block of* $\mathcal{P}$ *are in* $\mathcal{CD}$*-context with respect to* $\mathcal{A}^{\mathcal{P}}_{\mathrm{ext}}$*.*

We remark that neither $\mathcal{CD}$-strong equivalence nor $\mathcal{CD}$–context alone assure the lumpability of the underlying ODE system. Indeed, $\mathcal{CD}$-context overlooks transition rates, despite they are crucial for lumpability. To see that $\mathcal{CD}$-strong equivalence alone does not guarantee the lumpability of the ODEs is less obvious, and thus we provide the following example.

Running example (step 9/10). Let $\mathcal{M}_C \triangleq U_1 \parallel^{\min}_{\{\alpha\}} U_1'$, with U_1 defined as in *step 5/10*, and U_1' isomorphic to U_1, i.e., $U_1 \stackrel{def}{=} (\alpha,l).U_2$, $U_2 \stackrel{def}{=} (\gamma,s).U_1$ and $U_1' \stackrel{def}{=} (\alpha,l).U_2'$, $U_2' \stackrel{def}{=} (\gamma,s).U_1'$. Consider the equivalence relation $\mathcal{R}$ inducing the partition $\mathcal{P} = \{\{U_1,U_1'\},\{U_2,U_2'\}\}$. It can be shown that $\mathcal{R}$ is a $\mathcal{CD}$-strong equivalence. However, U_1 and U_1' are not in $\mathcal{CD}$–context with respect to the set $\mathcal{A}^{\mathcal{P}}_{\mathrm{ext}} = \{\alpha,\gamma\}$, in fact $\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\mathrm{ext}}}(U_1,\mathcal{M}_C) = \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\mathrm{ext}}}(U_1',\mathcal{M}_C) = \{\alpha\}$. Given that the ODEs of U_1 and U_1' are $\dot{\nu}_{U_1} = -\min(l\nu_{U_1}, l\nu_{U_1'}) + s\nu_{U_2}$ and $\dot{\nu}_{U_1'} = -\min(l\nu_{U_1}, l\nu_{U_1'}) + s\nu_{U_2'}$, the ODEs of $\mathcal{M}_C$ are not lumpable according to $\mathcal{P}$, as it is not possible to express $\dot{\nu}_{U_1} + \dot{\nu}_{U_1'}$ in terms of the aggregated variable $\nu_{U_1} + \nu_{U_1'}$ only, due to the presence of the term $\min(l\nu_{U_1}, l\nu_{U_1'})$. □

5 Congruent Differential Ordinary Lumpability

DOL exploits information regarding the whole structure of a FEPA model (and of a partition of its local states) to capture symmetries amongst its local states. Its ability to treat certain actions uniformly, e.g., as in condition (*iii*), comes at the price of not allowing compositional reasoning. Let us provide an example showing that DOL is not a congruence with respect to parallel composition.

Running example (step 10/10). Consider the FEPA model $\mathcal{M}_{RE}$, its partition $\mathcal{P}_{RE}$ of *step 4/10*, and the fluid atom U_1 defined in *step 5/10* as $U_1 \stackrel{def}{=} (\alpha, l).U_2$, $U_2 \stackrel{def}{=} (\gamma, s).U_1$, and its partition $\mathcal{P}_U = \{\{U_1\}, \{U_2\}\}$, where α is a $\mathcal{P}_U$-external transition. It can be shown that $\mathcal{P}_{RE}$ and $\mathcal{P}_U$ are both DOLPs. As regards $\mathcal{P}_{RE}$, we recall that P_2 and P_3 are in the same block, and although α belongs to the $\mathcal{CD}$ of P_2 but not to that of P_3, $\mathcal{P}_{RE}$ is a DOLP because α is $\mathcal{P}_{RE}$-internal. If we define the model $\mathcal{M}_{URE} = U_1 \parallel^{\mathcal{H}}_{\{\alpha\}} \mathcal{M}_{RE}$, and its partition $\mathcal{P}_{URE} = \mathcal{P}_U \cup \mathcal{P}_{RE}$, we have that α is $\mathcal{P}_{URE}$-external, and thus $\mathcal{P}_{URE}$ is not a DOLP of $\mathcal{M}_{URE}$. □

Herein, we introduce a more discriminating variant of DOL, where we neither distinguish between independent and current dependent actions, nor between internal and external actions. Such variant turns out to be a congruence with respect to the interaction operator. For this reason we name it *congruent differential ordinary lumpability* (CoDOL). Furthermore, in analogy to what done for DOL, we also provide a characterisation of CoDOL in terms of two symmetries (a local and a global one) among the local states of a FEPA model.

Definition 14 (Congruent differential ordinary lumpability). *Let $\mathcal{M}$ be a well-posed FEPA model. Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{M}) \times \mathcal{B}(\mathcal{M})$ be an equivalence relation over the local states of $\mathcal{B}(\mathcal{M})$, and $\mathcal{P}$ be the partition of $\mathcal{B}(\mathcal{M})$ induced by $\mathcal{R}$. We say that $\mathcal{R}$ is a* congruent differential ordinary lumpability *iff, for all $S \in \mathcal{P}$, whenever $P, Q \in S$, the two following conditions hold:*

(i) $\mathcal{A}(S) \triangleq \mathcal{A}(P) = \mathcal{A}(Q)$,

(ii) for all $\tilde{S} \in \mathcal{P}$, $\alpha \in \mathcal{A}(S)$, ν,

$$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) \; .$$

We refer to $\mathcal{P}$ as a congruent differential ordinary lumpable partition *(CoDOLP).*

Given that CoDOL treats all actions uniformly, the second condition essentially unifies (*ii*) and (*iii*) in Definition 11. CoDOL is a stronger variant of DOL.

Proposition 2. *Let $\mathcal{M}$ be a well-posed FEPA model, and $\mathcal{P}$ be a partition of $\mathcal{B}(\mathcal{M})$. If $\mathcal{P}$ is a congruent differential ordinary lumpable partition, then it is also a differential ordinary lumpable partition.*

The reverse implication does not hold. Consider, for instance, the DOLP $\mathcal{P}_{RE}$ of our running example. $\mathcal{P}_{RE}$ is not a CoDOLP, as it relates local states performing different actions, e.g., P_2 and P_3, where $\mathcal{A}(P_2) = \{\alpha, \gamma\}$ and $\mathcal{A}(P_3) = \{\xi\}$.

Finally, the next theorem provides the claimed congruence of CoDOL.

Theorem 3 (Congruence). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two FEPA models, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be congruent differential ordinary lumpable partitions of $\mathcal{B}(\mathcal{M}_1)$ and $\mathcal{B}(\mathcal{M}_2)$, respectively. Then, the partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is a congruent differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2)$, for any $L \subseteq \mathcal{A}$.*

Let us now turn to the characterisation of CoDOL. In this case, the local symmetry is actually captured by the notion of strong equivalence in [9], straightforwardly lifted to FEPA.

Definition 15 (Strong equivalence). *Let $\mathcal{M}$ be a well-posed FEPA model. Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{M}) \times \mathcal{B}(\mathcal{M})$ be an equivalence relation over $\mathcal{B}(\mathcal{M})$, and $\mathcal{P}$ be the partition of $\mathcal{B}(\mathcal{M})$ induced by $\mathcal{R}$. We say that $\mathcal{R}$ is a* strong equivalence *iff for all $S \in \mathcal{P}$, whenever $P, Q \in S$, the two following conditions hold:*

(i) $\mathcal{A}(S) \triangleq \mathcal{A}(P) = \mathcal{A}(Q)$,

(ii) for all $\tilde{S} \in \mathcal{P}$, $\alpha \in \mathcal{A}(S)$, $\qquad q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$.

Furthermore we say that P *and* Q *are* strong equivalent *if there exists a strong equivalence relating them.*

Unlike CoDOL, strong equivalence disregards the model influence, which is instead captured by the following notion of *congruent* $\mathcal{CD}$*-context.*

Definition 16 (Congruent $\mathcal{CD}$-context)**.** *Let* $\mathcal{M}$ *be a well-posed FEPA model. Let* $P, Q \in \mathcal{B}(\mathcal{M})$*. We say that* P*,* Q *are in* congruent $\mathcal{CD}$–context *iff* $\mathcal{A}(P) = \mathcal{A}(Q)$ *and one of the two following conditions hold:*

(i) it does not exist any occurrence $\overline{\mathcal{M}} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$ *within* $\mathcal{M}$ *with* $P \in \mathcal{B}(\mathcal{M}_1)$*, and* $Q \in \mathcal{B}(\mathcal{M}_2)$ *(or vice versa), or*

(ii) if such occurrence exists, then $\mathcal{CD}(P, \overline{\mathcal{M}}) = \mathcal{CD}(Q, \overline{\mathcal{M}}) = \emptyset$*.*

Lastly, the next theorem gives the characterisation of CoDOL.

Theorem 4. *Let* $\mathcal{M}$ *be a well-posed FEPA model and* $\mathcal{P}$ *a partition of* $\mathcal{B}(\mathcal{M})$*.* $\mathcal{P}$ *is congruent differential ordinary lumpable iff there exists a strong equivalence inducing* $\mathcal{P}$*, and the local states of each block of* $\mathcal{P}$ *are in congruent* $\mathcal{CD}$*-context.*

6 Related Work

This paper is most closely related to [17], where *exact fluid lumpability* for Markovian process algebra is introduced. Although our calculus is based on the *Fluid Process Algebra* of [17] and the motivation—ODE aggregation—is the same, the theory is much different. In [17] the unit of aggregation is the whole fluid atom, and has been motivated to capture symmetries arising from replicating many composite processes [19]. Here, instead, the aggregation is carried out at the finest level of detail of the fluid semantics, defining an equivalence relation over local states. Thus, unlike [17], DOL allows to collapse local states belonging to the same or distinct atoms. In addition, in our setting, a weaker notion of strong equivalence is a necessary condition for two local states to be related. Instead, in [17] a stronger stochastic characterisation holds whereby two related fluid atoms must be isomorphic. Furthermore, the underlying mathematics of the ODE aggregation is different. While in [17] two related fluid atoms are shown to have the same ODE solutions when provided with the same initial conditions, here, as discussed, the sum of the individual solutions of a block is equal to the solution of the corresponding aggregated ODE, independently on the initial conditions.

This approach is taken in [6] for model reduction of ODE models of biochemical reaction networks specified with the κ-calculus [7]. However the target language is different; κ is a rule-based language with a differential semantics with dynamics based on the law of mass action, at the core of chemical reactions. Our calculus, which allows to describe chemical reactions as well (resorting to $\mathcal{H} = \cdot$), is instead process-based, with the rule of interaction that is implicit in the compositional structure. In this respect, FEPA bears more resemblance to existing process algebra such as BioPEPA [5] or Cardelli's stochastic interacting processes [4]. Furthermore, we also support a synchronisation semantics based on capacity-sharing arguments (resorting to $\mathcal{H} = \min$), as in PEPA [9]. This can encode certain computer and communication networks (e.g., queueing networks) but clearly cannot be expressed as a chemical reaction.

Outside process algebra, the concurrency theoretical notions of equivalence have been related to abstractions of dynamical systems for continuous, discrete, and hybrid state spaces (e.g., [14, 15, 1]). There, behavioural relations are established directly at the level of the underlying mathematics. In the case of ODEs, they operate on a labelled transition system with infinite (continuous) state space, where each state denotes a possible solution of the ODE system at a specified time point. Instead, our relations are given at the model specification language level, which maintains a discrete state space given by associating a state with each ODE. Thus, these abstractions should be seen as complementary to our working definition of ODE aggregation [16], and can be seen to be equivalent when considered for the nonlinear, continuous, and autonomous ODE systems such as those induced by FEPA.

7 Conclusion

Differential ordinary lumpability is an equivalence relation for process algebra equipped with fluid semantics which induces an aggregation at the level of the underlying system of ordinary differential equations. It has been developed in a way that is conceptually similar to process-algebraic behavioural relations that give rise to a lumped continuous-time Markov chain, if a stochastic semantics is employed. We presented two variants that trade increased coarsening capability, via less discriminating power, for congruence, which allows compositional applications of aggregations. As with all analogous results available in the literature, differential ordinary lumpability gives only sufficient conditions for aggregation. Although proving necessity eluded us, we were not able to find examples of aggregations of systems specified in Fluid Extended Process Algebra which are not characterised by differential ordinary lumpability (in its weaker version). This remains an interesting open problem, which we conjecture might also help find a complete characterisation of aggregation in the Markovian semantics.

Acknowledgement This work was partially supported by the DFG project FEMPA, TR 1120/1-1, and by the EU project QUANTICOL, 600708.

References

[1] Alur, R., Henzinger, T., Lafferriere, G., Pappas, G.: Discrete abstractions of hybrid systems. Proceedings of the IEEE 88(7), 971–984 (July 2000)

[2] Buchholz, P.: Exact and Ordinary Lumpability in Finite Markov Chains. Journal of Applied Probability 31(1), 59–75 (1994)

[3] Buchholz, P.: Markovian Process Algebra: Composition and Equivalence. In: Proc. 2nd Workshop on Process Algebra and Performance Modelling. Erlangen, Germany (1994)

[4] Cardelli, L.: On process rate semantics. Theor. Comput. Sci. 391(3), 190–215 (2008)

[5] Ciocchetta, F., Hillston, J.: Bio-PEPA: A framework for the modelling and analysis of biological systems. Theor. Comput. Sci. 410(33-34), 3065–3084 (2009)

[6] Danos, V., Feret, J., Fontana, W., Harmer, R., Krivine, J.: Abstracting the differential semantics of rule-based models: Exact and automated model reduction. In: LICS. pp. 362–381. IEEE Computer Society (2010)

[7] Danos, V., Laneve, C.: Formal molecular biology. Theoretical Computer Science 325(1), 69–110 (2004)

[8] Hermanns, H., Ribaudo, M.: Exploiting symmetries in stochastic process algebras. In: European Simulation Multiconference. pp. 763–770. Manchester, UK (June 1998)

[9] Hillston, J.: A Compositional Approach to Performance Modelling. Cambridge University Press (1996)

[10] Hillston, J.: Fluid flow approximation of PEPA models. In: Proceedings of the Second International Conference on the Quantitative Evaluation of Systems. pp. 33–43. IEEE Computer Society Press, Torino, Italy (Sep 2005)

[11] Kemeny, J., Snell, J.: Finite Markov Chains. Springer New York, Heidelberg, Berlin (1976)

[12] Kwiatkowski, M., Stark, I.: The continuous pi-calculus: A process algebra for biochemical modelling. In: Proceedings of Computational Methods in Systems Biology. pp. 103–122. No. 5307 in LNCS (2008)

[13] Larsen, K.G., Skou, A.: Bisimulation through probabilistic testing. Information and Computation 94(1), 1–28 (1991)

[14] Pappas, G.J.: Bisimilar linear systems. Automatica 39(12), 2035–2047 (2003)

[15] van der Schaft, A.J., Schumacher, J.M.: Compositionality issues in discrete, continuous, and hybrid systems. International Journal of Robust and Nonlinear Control 11(5), 417–434 (2001)

[16] Toth, J., Li, G., Rabitz, H., Tomlin, A.S.: The effect of lumping and expanding on kinetic differential equations. SIAM Journal on Applied Mathematics 57(6), 1531–1556 (1997)

[17] Tschaikowski, M., Tribastone, M.: Exact fluid lumpability for Markovian process algebra. In: CONCUR. pp. 380–394. LNCS (2012)

[18] Tschaikowski, M., Tribastone, M.: Exact fluid lumpability in Markovian process algebra. Theoretical Computer Science (2013)

[19] Tschaikowski, M., Tribastone, M.: Tackling continuous state-space explosion in a Markovian process algebra. Theoretical Computer Science (2013)

A Preliminary Results

In this appendix we collect a number of preliminary results which are instrumental to prove the main results presented in the paper.

We start with Proposition 1, given in Section 3.

Proposition 1. *Let $\mathcal{M}$ be a FEPA model, $P \in \mathcal{B}(\mathcal{M})$ and $\alpha \in \mathcal{A}$. Then, $\alpha \notin \mathcal{D}(P, \mathcal{M}) \Leftrightarrow \mathcal{F}_\alpha(\mathcal{M}, \nu, P) = 1$, for any ν.*

Proof. The direction $\Longrightarrow$ follows directly from Definition 10 and Definition 8.

To prove the implication $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = 1\ , \forall \nu \ \Rightarrow \ \alpha \notin \mathcal{D}(P, \mathcal{M})$ we proceed, instead, by contradiction. Let us assume towards a contradiction that there exists an $\alpha \in \mathcal{D}(P, \mathcal{M})$ such that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = 1$ for all ν. The assumption $\alpha \in \mathcal{D}(P, \mathcal{M})$ implies that there exists at least an occurrence $\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$ within $\mathcal{M}$, with $\alpha \in L$, and P is either in $\mathcal{B}(\mathcal{M}_1)$ or $\mathcal{B}(\mathcal{M}_2)$. We assume, without loss of generality, that $P \in \mathcal{B}(\mathcal{M}_1)$. Therefore, from Definition 10, we can infer that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P)$ will be proportional to $\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)}$. Choosing ν such that $r_\alpha(\mathcal{M}_2, \nu) = 0$ (it always exists!) assures that $r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu) = 0$; hence, we have found a population function ν such that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) \neq 1$, which leads us to contradiction and concludes the proof. □

The following proposition states that if a model is well-posed, then all its interacting sub-models are well-posed.

Proposition 3. *Let $\mathcal{M}$ be a well-posed model, then for every occurrence of $\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$ in $\mathcal{M}$, we have that $\mathcal{M}_1$ and $\mathcal{M}_2$ are well-posed.*

Proof. By case distinction on the grammar.

- $\mathcal{M} = P$: This case is trivial, as a fluid atom does not have any occurrence of the interaction operator, and thus it is well-posed by definition.
- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$: This case is proved by contradiction. Let us assume that $\mathcal{M}_1$ ($\mathcal{M}_2$) is ill-posed, i.e. that it has an occurrence $\mathcal{M}'_1 \parallel^{\mathcal{H}}_{L'} \mathcal{M}'_2$ not satisfying the condition of well-posedness. Clearly, given that $\mathcal{M}'_1 \parallel^{\mathcal{H}}_{L'} \mathcal{M}'_2$ occurs in a sub-model of $\mathcal{M}$, we also have that it occurs in $\mathcal{M}$, implying that $\mathcal{M}$ is ill-posed, obtaining thus a contradiction.

□

The well-posedness assumption, introduced in Section 2, has an interesting and quite intuitive relation with the model influence: it assures the existence of a population function for which the model cannot impede a local state from performing an action.

Proposition 4. *Let $\mathcal{M}$ be a well-posed model, $\alpha \in \mathcal{A}$ and $P' \in \mathcal{B}(\mathcal{M})$. Then there exists a population function ν such that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P') > 0$.*

Proof. The proof proceeds by structural induction on the FEPA model $\mathcal{M}$.

- $\mathcal{M} = P$: This case is trivial, as by Definition 10 we have that $\mathcal{F}_\alpha(P, \nu, P') = 1$ for any $P' \in \mathcal{B}(P)$ and any population function.
- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$: We can have that either $P' \in \mathcal{B}(\mathcal{M}_1)$ or $P' \in \mathcal{B}(\mathcal{M}_2)$. Without loss of generality we can assume the first case. Moreover, by Proposition 3 we know that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are well-posed, and thus the I.H. can be applied on them. By Definition 10 we have:

- $\alpha \notin L$: $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu, P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P')$. By applying the I.H. on $\mathcal{M}_1$ we know that there exists a population function ν_1 such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') > 0$. Thus, for any population function $\nu' = (\nu_1, \nu_2)$, where ν_2 is defined for the components in $\mathcal{M}_2$, we have that $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu', P') > 0$.

- $\alpha \in L$: $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu, P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P') \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu))}{r_\alpha(\mathcal{M}_1, \nu)}$. Let us consider ν as (ν_1, ν_2), where ν_1 and ν_2 are defined for $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. We have to find a ν_1 such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') > 0$, and $r_\alpha(\mathcal{M}_1, \nu_1) > 0$, and a ν_2 such that $r_\alpha(\mathcal{M}_2, \nu_2) > 0$. Given that $\mathcal{M}$ is well-posed and $\alpha \in L$, we know that there exist at least two population functions ν_1', ν_2' such that $r_\alpha(\mathcal{M}_1, \nu_1') > 0$ and $r_\alpha(\mathcal{M}_2, \nu_2') > 0$. We can thus fix $\nu_2 = \nu_2'$. By applying the I.H. on $\mathcal{M}_1$, we know that there exists a population function ν_1'' such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1'', P') > 0$, and thus we can apply Lemma 4 to $\mathcal{M}_1$. Two cases can arise from Lemma 4. The first case is $\mathcal{F}_\alpha(\mathcal{M}_1, \hat{\nu}_1, P') = 1$ for any $\hat{\nu}_1$, and thus also for the ν_1' chosen such that $r_\alpha(\mathcal{M}_1, \nu_1') > 0$. Thus we have that $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, (\nu_1', \nu_2'), P') > 0$, obtaining the claim. The second case arising from Lemma 4 is that for any $K > 0$ there exists a ν_K such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_K, P') = 1$, and $r_\alpha(\mathcal{M}_1, \nu_K) = K$. Thus, for any $K > 0$ we have that $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, (\nu_K, \nu_2'), P') > 0$, obtaining the claim.

□

The next proposition provides an inclusion property of the restricted current dependent action set.

Proposition 5. *Let $\mathcal{M}$ be a FEPA model, let $\mathcal{M}'$ be a sub-model of $\mathcal{M}$, and let $\hat{\mathcal{A}}$ be a set of actions. Then for any $P \in \mathcal{B}(\mathcal{M}')$, $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}') \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$.*

Proof. We prove the claim by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: The claim follows by noticing that the only sub-model of P is P itself.
- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$: We assume, without loss of generality, $P \in \mathcal{B}(\mathcal{M}_1)$. We notice that in order to prove the claim it suffices to show that, given a set of actions $\hat{\mathcal{A}}$, it holds $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$. In fact, by I.H. we know that for any sub-model $\mathcal{M}'$ of $\mathcal{M}_1$ such that $P \in \mathcal{B}(\mathcal{M}')$ we have $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}') \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$.

  The proof of the inclusion $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ relies on standard set theory and is given below.

$$\begin{aligned} \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) &= \mathcal{D}(P, \mathcal{M}) \cap \mathcal{A}(P) \cap \hat{\mathcal{A}} \\ &= (\mathcal{D}(P, \mathcal{M}_1) \cup L) \cap \mathcal{A}(P) \cap \hat{\mathcal{A}} \\ &= (\mathcal{D}(P, \mathcal{M}_1) \cap \mathcal{A}(P) \cap \hat{\mathcal{A}}) \cup (L \cap \mathcal{A}(P) \cap \hat{\mathcal{A}}) \\ &= \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \cup (L \cap \mathcal{A}(P) \cap \hat{\mathcal{A}}) \ . \end{aligned}$$

□

Proposition 6. *Let $\mathcal{M}$ be a FEPA model and $\hat{\mathcal{A}}$ be a set of actions. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be such that they are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$. For any sub-model $\mathcal{M}'$ of $\mathcal{M}$ such that $P, Q \in \mathcal{B}(\mathcal{M}')$ it holds that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}') = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}')$.*

Proof. Let us assume towards a contradiction that there exists a sub-model $\mathcal{M}'$ of $\mathcal{M}$ with $P, Q \in \mathcal{B}(\mathcal{M}')$ such that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}') \neq \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}')$. Thus, without loss of generality, we can assume that there exists an action α such that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}')$ and $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}')$. Exploiting Proposition 5 and the assumption that P, Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ (in particular, $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$) we can infer that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$. Therefore, by Definition 8 we have that $\alpha \in \hat{\mathcal{A}} \cap \mathcal{A}(P)$ and $\alpha \in \hat{\mathcal{A}} \cap \mathcal{A}(Q)$. If $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}')$ and $\alpha \in \hat{\mathcal{A}} \cap \mathcal{A}(Q)$ then $\alpha \notin \mathcal{D}(Q, \mathcal{M}')$ whereas, if $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}')$, then $\alpha \in \mathcal{D}(P, \mathcal{M}')$.

The fact that $\alpha \in \mathcal{D}(P, \mathcal{M}')$ and $\alpha \notin \mathcal{D}(Q, \mathcal{M}')$ assures the existence of an occurrence $\bar{\mathcal{M}} = \mathcal{M}'_1 \parallel^{\mathcal{H}}_L \mathcal{M}'_2$ within $\mathcal{M}'$ such that $P \in \mathcal{B}(\mathcal{M}'_1)$, $Q \in \mathcal{B}(\mathcal{M}'_2)$, $\alpha \notin L$ and such that $\alpha \in \mathcal{D}(P, \mathcal{M}'_1)$ while $\alpha \notin \mathcal{D}(Q, \mathcal{M}'_2)$. However, knowing that $\alpha \in \hat{\mathcal{A}} \cap \mathcal{A}(P)$ and $\alpha \in \hat{\mathcal{A}} \cap \mathcal{A}(Q)$ the above would imply $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \bar{\mathcal{M}})$ whilst $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \bar{\mathcal{M}})$, contradicting the assumption that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$. In fact, being P and Q in $\mathcal{CD}$–context, we should have $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \bar{\mathcal{M}}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \bar{\mathcal{M}}) = \emptyset$. □

Given a model $\mathcal{M}$, the next proposition states that the notion of $\mathcal{CD}$–context with respect to a set of actions is preserved while descending the syntax tree of the model.

Proposition 7. *Let $\mathcal{M}$ be a FEPA model, and $\hat{\mathcal{A}}$ be a set of actions. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be such that they are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}$. Then for any sub-model $\mathcal{M}'$ of $\mathcal{M}$ such that $P, Q \in \mathcal{B}(\mathcal{M}')$, P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}'$ as well.*

Proof. We have to prove that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}') = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}')$, and that either condition *i)* or *ii)* of Definition 13 hold for $\mathcal{M}'$. The equivalence of the $\hat{\mathcal{A}}$-restricted current dependent contexts follows directly from Proposition 6. As regards condition *i)* or *ii)* of Definition 13, if $\overline{\mathcal{M}}$ does not occur in $\mathcal{M}$, then neither it occurs in $\mathcal{M}'$. If instead $\overline{\mathcal{M}}$ occurs in $\mathcal{M}$, then the fact that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}$, and that $P, Q \in \mathcal{B}(\mathcal{M}')$ implies that $\overline{\mathcal{M}}$ must be an occurrence within $\mathcal{M}'$, with $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \overline{\mathcal{M}}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \overline{\mathcal{M}}) = \emptyset$. □

The first lemma we present allows us identifying the contribution that local states yield to the rate of the whole model for those actions for which the local states behave independently.

Lemma 1. *Let $\mathcal{M}$ be a FEPA model. Let $K \subseteq \mathcal{B}(\mathcal{M})$, and α an action such that $\alpha \notin \mathcal{CD}(K, \mathcal{M})$. Then, for any ν,*

$$r_\alpha(\mathcal{M}, \nu) = \sum_{P \in K} r_\alpha(P)\nu_P \ + \ r_\alpha(\mathcal{M}, \nu^K) ,$$

where ν^K is defined as $\nu^K_P = \nu_P$ if $P \notin K$ and $\nu^K_P = 0$ if $P \in K$.

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: By Definition 3, for any α we have that

$$r_\alpha(P, \nu) = \sum_{P' \in \mathcal{B}(P)} r_\alpha(P')\nu_{P'} .$$

  For any $K \subseteq \mathcal{B}(P)$, the above summation can be rewritten as

$$\sum_{P' \in \mathcal{B}(P) \cap K} r_\alpha(P')\nu_{P'} + \sum_{P' \in \mathcal{B}(P) \setminus K} r_\alpha(P')\nu_{P'} + \left(\sum_{P' \in K} r_\alpha(P') \cdot 0 \right) ,$$

  which is equal to $\sum_{P' \in K} r_\alpha(P')\nu_{P'} + r_\alpha(P, \nu^K)$, closing the case.

- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$: Let $K \subseteq \mathcal{B}(\mathcal{M})$, and $\alpha \notin \mathcal{CD}(K, \mathcal{M})$. We have to distinguish among two cases: $\alpha \in L$ and $\alpha \notin L$.

- $\alpha \in L$: By Definition 3 we have $r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu) = \mathcal{H}(r_\alpha(\mathcal{M}_1, \nu), r_\alpha(\mathcal{M}_2, \nu))$. Note that the α-apparent rate in $\mathcal{M}_1$ does not depend on the population of the local states of $\mathcal{B}(\mathcal{M}_2)$ (and vice versa), which can thus be freely modified without affecting the α-apparent rate in $\mathcal{M}_1$.

Given that $\alpha \in L$, for all $P \in \mathcal{B}(\mathcal{M})$ we have that $\alpha \in \mathcal{D}(P, \mathcal{M})$. From the assumption we know that for all $P \in K$, $\alpha \notin \mathcal{CD}(P, \mathcal{M})$, i.e. $\alpha \notin \mathcal{D}(P, \mathcal{M}) \cap \mathcal{A}(P)$, implying that $\alpha \notin A(P)$. Therefore, we have $r_\alpha(P) = 0$ for all $P \in K$, as well as $\sum_{P \in K} r_\alpha(P)\nu_P = 0$. Let us define $K_i = K \cap \mathcal{B}(\mathcal{M}_i)$ for $i \in \{1, 2\}$. We focus on $\mathcal{M}_1$, but similar arguments hold for $\mathcal{M}_2$. Given that $K = K_1 \cup K_2$ and $\alpha \notin \mathcal{CD}(K_1 \cup K_2, \mathcal{M})$, we have $\alpha \notin \mathcal{CD}(K_1, \mathcal{M})$. Note that $\mathcal{CD}^{\hat{\mathcal{A}}}(K_1, \mathcal{M}) = \mathcal{CD}(K_1, \mathcal{M})$, if we set $\hat{\mathcal{A}} = \mathcal{A}$. Thus, we can apply Proposition 5 to any $P \in K_1$, obtaining $\mathcal{CD}(K_1, \mathcal{M}_1) \subseteq \mathcal{CD}(K_1, \mathcal{M})$, which in turn implies $\alpha \notin \mathcal{CD}(K_1, \mathcal{M}_1)$. This allows us to apply the I.H. to $\mathcal{M}_1$, obtaining $r_\alpha(\mathcal{M}_1, \nu) = \sum_{P \in K_1} r_\alpha(P)\nu_P \ + \ r_\alpha(\mathcal{M}_1, \nu^{K_1})$, which, given that $r_\alpha(P) = 0$ for any $P \in K$, and $K_1 \subseteq K$, is equal to $r_\alpha(\mathcal{M}_1, \nu^{K_1})$. Moreover, given that the α-apparent rate of $\mathcal{M}_1$ does not depend on the population of the local states in $\mathcal{B}(\mathcal{M}_2)$, we can write $r_\alpha(\mathcal{M}_1, \nu) = r_\alpha(\mathcal{M}_1, \nu^K)$. Applying similar arguments to $\mathcal{M}_2$ we obtain $r_\alpha(\mathcal{M}_2, \nu_2) = r_\alpha(\mathcal{M}_2, \nu_2^K)$. Finally, we conclude that $r_\alpha(\mathcal{M}, \nu) = \mathcal{H}(r_\alpha(\mathcal{M}_1, \nu^K), r_\alpha(\mathcal{M}_2, \nu^K)) = r_\alpha(\mathcal{M}, \nu^K) = \sum_{P \in K} 0 \cdot \nu_P \ + \ r_\alpha(\mathcal{M}, \nu^K) = \sum_{P \in K} r_\alpha(P)\nu_P \ + \ r_\alpha(\mathcal{M}, \nu^K)$.

- $\alpha \notin L$: By Definition 3 we know that $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu) = r_\alpha(\mathcal{M}_1, \nu) + r_\alpha(\mathcal{M}_2, \nu)$. We focus on $\mathcal{M}_1$, but similar arguments hold for $\mathcal{M}_2$. Similarly to the previous case ($\alpha \in L$), we have $\alpha \notin \mathcal{CD}(K_1, \mathcal{M}_1)$. This allows us to apply the I.H. to $\mathcal{M}_1$, obtaining $r_\alpha(\mathcal{M}_1, \nu) = \sum_{P \in K_1} r_\alpha(P)\nu_P + r_\alpha(\mathcal{M}_1, \nu^{K_1})$. Moreover, given that the α-apparent rate of $\mathcal{M}_1$ does not depend on the population of the local states in $\mathcal{B}(\mathcal{M}_2)$, we can write

$$r_\alpha(\mathcal{M}_1, \nu) = \sum_{P \in K_1} r_\alpha(P)\nu_P \ + \ r_\alpha(\mathcal{M}_1, \nu^K).$$

Similar arguments can be applied to $\mathcal{M}_2$, yielding

$$\begin{aligned} r_\alpha(\mathcal{M}, \nu) &= \sum_{P \in K_1} r_\alpha(P)\nu_P + \sum_{P \in K_2} r_\alpha(P)\nu_P + r_\alpha(\mathcal{M}_1, \nu^K) + r_\alpha(\mathcal{M}_2, \nu^K) \\ &= \sum_{P \in K} r_\alpha(P)\nu_P + r_\alpha(\mathcal{M}, \nu^K) \ , \end{aligned}$$

concluding the proof.

□

The lemma below is similar in nature to Lemma 1 but, instead of the apparent rate, pertains the model influence. It says that local states which behave independently with respect to a certain action, have no effect whatsoever on the influence that the model exerts through that action on its local states.

Lemma 2. *Let $\mathcal{M}$ be a FEPA model. Let $K \subseteq \mathcal{B}(\mathcal{M})$, and α an action such that $\alpha \notin \mathcal{CD}(K, \mathcal{M})$. Then, for any $P \in \mathcal{B}(\mathcal{M})$, and for any ν,*

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) \ ,$$

where ν^K is defined as $\nu_P^K = \nu_P$ if $P \notin K$ and $\nu_P^K = 0$ if $P \in K$.

Proof. We proceed by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: Firstly, we remark that for all $K \subseteq \mathcal{B}(P)$ we have that $\mathcal{CD}(K, P) = \emptyset$. Thus, the claim has to be proved for any α. The claim holds, however, by noticing that for any α, for any $P' \in \mathcal{B}(P)$, $\mathcal{F}_\alpha(P, \nu, P') = 1$ for any ν, and thus also for ν^K.

- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: Let $K \subseteq \mathcal{B}(\mathcal{M})$, α be such that $\alpha \notin \mathcal{CD}(K, \mathcal{M})$ and, without loss of generality, $P \in \mathcal{B}(\mathcal{M}_1)$. We have to distinguish amongst two cases: $\alpha \in L$, $\alpha \notin L$.

- $\alpha \in L$: By Definition 10, we have

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} .$$

Given that $\alpha \in L$, we know that for any $P \in \mathcal{B}(\mathcal{M})$, $\alpha \in \mathcal{D}(P, \mathcal{M})$. The assumption $\alpha \notin \mathcal{CD}(K, \mathcal{M})$, implies that $\alpha \notin \mathcal{A}(P) \cap \mathcal{D}(P, \mathcal{M})$ for any $P \in K$, and thus $r_\alpha(P) = 0$ for any $P \in K$. Let us denote $K_i = K \cap \mathcal{B}(\mathcal{M}_i)$, $i \in \{1, 2\}$. Given that $K = K_1 \cup K_2$, the assumption $\alpha \notin \mathcal{CD}(K_1 \cup K_2, \mathcal{M})$ implies that $\alpha \notin \mathcal{CD}(K_i, \mathcal{M})$, $i \in \{1, 2\}$. Moreover, note that $\mathcal{CD}^{\hat{\mathcal{A}}}(K_i, \mathcal{M}) = \mathcal{CD}(K_i, \mathcal{M})$, if we set $\hat{\mathcal{A}} = \mathcal{A}$. Thus, we can apply Proposition 5 to any $P \in K_i$, obtaining $\mathcal{CD}(K_i, \mathcal{M}_i) \subseteq \mathcal{CD}(K_i, \mathcal{M})$, assuring that $\alpha \notin \mathcal{CD}(K_i, \mathcal{M}_i)$. We can thus apply Lemma 1, together with the above-remarked fact that $r_\alpha(P) = 0$ for any $P \in K$, to obtain $r_\alpha(\mathcal{M}_i, \nu) = r_\alpha(\mathcal{M}_i, \nu^{K_i})$, for $i \in \{1, 2\}$. We also point out that $r_\alpha(\mathcal{M}_1, \nu)$ (resp. $r_\alpha(\mathcal{M}_2, \nu)$) does not depend on the population assigned to local states in $\mathcal{B}(\mathcal{M}_2)$ (resp. $\mathcal{B}(\mathcal{M}_1)$); thence,

$$\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu^K)}{r_\alpha(\mathcal{M}_1, \nu^K)} .$$

As far as $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$ is concerned, the fact that $K_1 \subseteq \mathcal{B}(\mathcal{M}_1)$, $\alpha \notin \mathcal{CD}(K_1, \mathcal{M}_1)$ and $P \in \mathcal{B}(\mathcal{M}_1)$ allows us to apply the I.H. obtaining

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu^{K_1}, P) .$$

On the other hand, the function $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$ does not depend on the population function assigned to local states in $\mathcal{B}(\mathcal{M}_2)$, and thus $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^{K_1}, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P)$. This concludes the proof for the case $\alpha \in L$.

- $\alpha \notin L$: We recall that we are assuming $K \subseteq \mathcal{B}(\mathcal{M})$, α be such that $\alpha \notin \mathcal{CD}(K, \mathcal{M})$ and $P \in \mathcal{B}(\mathcal{M}_1)$. By Definition 10, we have

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) .$$

As done in the previous case, we denote $K_i = K \cap \mathcal{B}(\mathcal{M}_i)$, $i \in \{1, 2\}$. Given that $K = K_1 \cup K_2$, the assumption $\alpha \notin \mathcal{CD}(K_1 \cup K_2, \mathcal{M})$ implies that $\alpha \notin \mathcal{CD}(K_i, \mathcal{M})$, $i \in \{1, 2\}$, and thus $\alpha \notin \mathcal{CD}(K_i, \mathcal{M}_i)$. The fact that $K_1 \subseteq \mathcal{B}(\mathcal{M}_1)$, $\alpha \notin \mathcal{CD}(K_1, \mathcal{M}_1)$ and $P \in \mathcal{B}(\mathcal{M}_1)$ allows us to apply the I.H. obtaining

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu^{K_1}, P) .$$

The independence of $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^{K_1}, P)$ from the population function assigned to local states in $\mathcal{B}(\mathcal{M}_2)$ leads us to $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^{K_1}, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P)$, which concludes the proof.

□

The next lemma states that if a FEPA model $\mathcal{M}$ is capable of performing an action α, i.e., there exists a population function ν such that $r_\alpha(\mathcal{M}, \nu) > 0$, then it is possible to tune the population within the model to make its α rate equal to any arbitrary non-negative value.

Lemma 3. *Let $\mathcal{M}$ be a FEPA model and $\alpha \in \mathcal{A}$ such that there exists a ν_α for which $r_\alpha(\mathcal{M}, \nu_\alpha) > 0$. Then, for any non-negative real value K there exists a function ν_K such that $r_\alpha(\mathcal{M}, \nu_K) = K$.*

Proof. The proof proceeds by structural induction on the FEPA model $\mathcal{M}$.

- $\mathcal{M} = P$: Let $\alpha \in \mathcal{A}$ be such that there exists a population function ν_α with $r_\alpha(P, \nu_\alpha) > 0$. By the base case of Definition 3, this implies the existence of at least one local state $P' \in \mathcal{B}(P)$ such that $r_\alpha(P') > 0$. Let K be any non-negative real number, then the population function ν assigning 0 to any $P'' \neq P'$ in $\mathcal{B}(P)$, and to P' the value $K/r_\alpha(P')$ yields the claim.

- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: Let $\alpha \in \mathcal{A}$ be such that there exists a population function ν_α with $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_\alpha) > 0$. We write $\nu_\alpha = (\nu_\alpha^1, \nu_\alpha^2)$ to emphasize the components of ν belonging to $\mathcal{M}_1$ and $\mathcal{M}_2$. We make a case distinction and make use of Definition 3.

- $\alpha \in L$: We first consider the case $\mathcal{H} = \min$, namely, $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_\alpha) = \min(r_\alpha(\mathcal{M}_1, \nu_\alpha), r_\alpha(\mathcal{M}_2, \nu_\alpha))$. Note that we can rewrite the right-hand-side of the latter equation as $\min(r_\alpha(\mathcal{M}_1, \nu_\alpha^1), r_\alpha(\mathcal{M}_2, \nu_\alpha^2))$ exploiting the fact that $r_\alpha(\mathcal{M}_1, \nu_\alpha)$ (resp. $r_\alpha(\mathcal{M}_2, \nu_\alpha)$) does not depend on ν_α^2 (resp. ν_α^1). The assumption $\min(r_\alpha(\mathcal{M}_1, \nu_\alpha^1), r_\alpha(\mathcal{M}_2, \nu_\alpha^2)) > 0$ implies that both arguments must be positive. By I.H. on $\mathcal{M}_1$ and $\mathcal{M}_2$ we can infer that for any $K_1, K_2 \geq 0$ there exist $\hat{\nu}_\alpha^1$, $\hat{\nu}_\alpha^2$ such that $r_\alpha(\mathcal{M}_1, \hat{\nu}_\alpha^1) = K_1$ and $r_\alpha(\mathcal{M}_2, \hat{\nu}_\alpha^2) = K_2$. Let now K be any non-negative real number, choosing $K_1 = K_2 = K$ yields the claim. The proof for the case $\mathcal{H} = \cdot$ follows the same lines with the unique difference that one has to chose $K_1 \cdot K_2 = K$, for instance $K_1 = 1$ and $K_2 = K$.

- $\alpha \notin L$: $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_\alpha) = r_\alpha(\mathcal{M}_1, \nu_\alpha^1) + r_\alpha(\mathcal{M}_2, \nu_\alpha^2)$ regardless of the function $\mathcal{H}$. The assumption $r_\alpha(\mathcal{M}_1, \nu_\alpha^1) + r_\alpha(\mathcal{M}_2, \nu_\alpha^2) > 0$ implies that at least one of the addends must be positive. Let us consider the case in which only one is positive, and, without loss of generality, we take $r_\alpha(\mathcal{M}_1, \nu_\alpha^1) > 0$ and $r_\alpha(\mathcal{M}_2, \nu_\alpha^2) = 0$. By I.H. on $\mathcal{M}_1$, we can infer that for any $K_1 > 0$ there exists a $\hat{\nu}_\alpha^1$ such that $r_\alpha(\mathcal{M}_1, \hat{\nu}_\alpha^1) = K_1$. Choosing $K_1 = K$ yields the claim. The case in which both addends are positive is similar to the case $\alpha \in L$, with the only difference that K_1, K_2 are chosen in such a way that their sum is equal to K.

□

The following result tells us that, given a FEPA model $\mathcal{M}$, a component $P \in \mathcal{B}(\mathcal{M})$ and an action α, if there exists a population such that $\mathcal{M}$ does not forbid P from executing α, then we have that two cases might hold: (1) the model never imposes any influence on the rate at which P executes α (e.g. α is not used for interaction in $\mathcal{M}$), (2) we have that for any $K > 0$ it is always possible to tune the population to simultaneously make unitary the model influence upon the rate at which P executes α, and to set to K the α-apparent rate of the model $\mathcal{M}$. We remark that the two above mentioned cases are not exclusive.

Lemma 4. *Let $\mathcal{M}$ be a FEPA model. Let $\alpha \in \mathcal{A}$ and $P' \in \mathcal{B}(\mathcal{M})$ be such that there exists a population function $\nu_{\alpha,P'}$ for which $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$. Then any of the two following cases holds:*

— for any population function ν

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P') = 1 \ ,$$

or

— for any $K > 0$ there exists a population function ν_K such that

$$\mathcal{F}_\alpha(\mathcal{M}, \nu_K, P') = 1 \qquad \textit{and} \qquad r_\alpha(\mathcal{M}, \nu_K) = K \ .$$

Proof. The proof proceeds by structural induction on the FEPA model $\mathcal{M}$.

- $\mathcal{M} = P$: By the base case of Definition 10, we know that $\mathcal{F}_\alpha(P, \nu, P') = 1$ for any ν. Moreover, if there exists at least a $\hat{P} \in \mathcal{B}(P)$ such that $r_\alpha(\hat{P}) > 0$, then by Definition 3 we know that for any population function $\hat{\nu}$ assigning a positive population to $\hat{P}$ we have $r_\alpha(P, \hat{\nu}) > 0$. For these cases we can thus apply Lemma 3, ensuring that we can find a ν_K such that $r_\alpha(P, \nu_K) = K$ (and $\mathcal{F}_\alpha(P, \nu_K, P') = 1$).

- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: Let $\alpha \in \mathcal{A}$ and $P' \in \mathcal{B}(\mathcal{M})$ be such that there exists a population function $\nu_{\alpha,P'}$ for which $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$. We assume, without loss of generality, that $P' \in \mathcal{B}(\mathcal{M}_1)$. We make a case distinction:

- $\alpha \in L$: We now prove that for $\alpha \in L$ we always have the second result, i.e. that for any $K > 0$ there exists a population function ν_K such that $\mathcal{F}_\alpha(\mathcal{M}, \nu_K, P') = 1$ and $r_\alpha(\mathcal{M}, \nu_K) = K$. By Definition 10 together with the assumption $P' \in \mathcal{B}(\mathcal{M}_1)$we have:

$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'}, P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'})}{r_\alpha(\mathcal{M}_1, \nu_{\alpha,P'})} .$$

The assumption $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$ assures that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') > 0$, $r_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}) > 0$, $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'}) > 0$, and thus also $r_\alpha(\mathcal{M}_2, \nu_{\alpha,P'}) > 0$. Note that this allows us to apply the I.H. on $\mathcal{M}_1$, as well as Lemma 3 to both $\mathcal{M}_1$ and $\mathcal{M}_2$.

By I.H. on $\mathcal{M}_1$ we know that two cases might happen: the first case is that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^1, P') = 1$ for any ν^1 on $\mathcal{B}(\mathcal{M}_1)$. When this holds, by exploiting Lemma 3 for both $\mathcal{M}_1$ and $\mathcal{M}_2$ we obtain that for any non-negative reals K_1, K_2, there exist two functions ν_{K_1}, ν_{K_2} such that $r_\alpha(\mathcal{M}_1, \nu_{K_1}) = K_1$, and $r_\alpha(\mathcal{M}_2, \nu_{K_2}) = K_2$. For any $K > 0$, by fixing $K_1 = K$, and by choosing a K_2 such that $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_{K_1}, \nu_{K_2})) = K$ (for instance, if $\mathcal{H} = \min$ we choose $K_2 = K + 1$, whereas if $\mathcal{H} = \cdot$, we choose $K_2 = 1$), we have that $r_\alpha(\mathcal{M}, (\nu_{K_1}, \nu_{K_2})) = K$ (and $\mathcal{F}_\alpha(\mathcal{M}, (\nu_{K_1}, \nu_{K_2}), P') = 1$).

The second case arising from the I.H. is that for any $K_1 > 0$ there exists a ν_{K_1} such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{K_1}, P') = 1$ and $r_\alpha(\mathcal{M}_1, \nu_{K_1}) = K_1$. By exploiting Lemma 3 for $\mathcal{M}_2$, for any $K_2 \geq 0$ we can choose a ν_{K_2} such that $r_\alpha(\mathcal{M}_2, \nu_{K_2}) = K_2$. For any $K > 0$, by fixing $K_1 = K$, and by choosing a K_2 such that $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_{K_1}, \nu_{K_2})) = K$, we have that $r_\alpha(\mathcal{M}, (\nu_{K_1}, \nu_{K_2})) = K$ (and $\mathcal{F}_\alpha(\mathcal{M}, (\nu_{K_1}, \nu_{K_2}), P') = 1$).

- $\alpha \notin L$: By Definition 10 we have: $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'}, P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P')$. The assumption $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$ assures that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') > 0$. By I.H. on $\mathcal{M}_1$ we know that two cases might happen: the first case is that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^1, P') = 1$ for any ν^1 on $\mathcal{B}(\mathcal{M}_1)$. Therefore, we have that $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu^1, \nu^2), P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P') = 1$ for any ν^1 on $\mathcal{B}(\mathcal{M}_1)$ and ν^2 on $\mathcal{B}(\mathcal{M}_2)$.

The second case arising from the I.H. is that for any $K_1 > 0$ there exists a ν_{K_1} such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{K_1}, P') = 1$ and $r_\alpha(\mathcal{M}_1, \nu_{K_1}) = K_1$. As far as $\mathcal{M}_2$ is concerned, either $r_\alpha(\mathcal{M}_2, \nu_2) = 0$ for any ν_2, or there exists at least a population function $\hat{\nu}_2$ such that $r_\alpha(\mathcal{M}_2, \hat{\nu}_2) > 0$. In the former case, following Definition 3, for any $K > 0$ by choosing $K_1 = K$ we have $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_{K_1}, \nu_2)) = r_\alpha(\mathcal{M}_1, \nu_{K_1}) = K$ and $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_{K_1}, \nu_2), P') = 1$. In the latter case, by exploiting Lemma 3 for $\mathcal{M}_2$, we have that for any $K_2 \geq 0$ we can choose a ν_{K_2} such that $r_\alpha(\mathcal{M}_2, \nu_{K_2}) = K_2$. Following Definition 3, for any $K > 0$, by choosing K_1 and K_2 such that $K_1 + K_2 = K$, we have that $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_{K_1}, \nu_{K_2})) = r_\alpha(\mathcal{M}_1, \nu_{K_1}) + r_\alpha(\mathcal{M}_2, \nu_{K_2}) = K$ and $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_{K_1}, \nu_{K_2}), P') = 1$.

□

The lemma below extends Lemma 4 allowing to simultaneously consider two components. Intuitively, it says that for any two components in a well-posed FEPA model, it is possible to find a population function which simultaneously makes the model influence upon the two components equal to 1; equivalently, there exists a population function ν for which the model exerts no influence whatsoever upon the components rates.

Lemma 5. *Let $\mathcal{M}$ be a FEPA model. Let $\alpha \in \mathcal{A}$, and $P', P'' \in \mathcal{B}(\mathcal{M})$ be such that there exist two population functions $\nu_{\alpha,P'}$ and $\nu_{\alpha,P''}$ for which it holds that $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$ and $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P''}, P'') > 0$, respectively. Then either of the two following cases holds:*

— *for any population function ν*

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, \nu, P'') ,$$

or

— *there exists a population function* $\bar{\nu}$ *such that*

$$\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P'') \qquad \textit{and} \qquad r_\alpha(\mathcal{M}, \bar{\nu}) = 1 \ .$$

Proof. Before starting the proof, we remark that the difference between the condition $r_\alpha(\mathcal{M}, \bar{\nu}) = 1$ and the condition $r_\alpha(\mathcal{M}, \bar{\nu}) = K$ given in the previous lemmas is not accidental. As it turns out, the statement cannot be true with $K \neq 1$ for the interaction function $\mathcal{H} = \cdot$. The proof proceeds by structural induction on the FEPA model $\mathcal{M}$.

- $\mathcal{M} = P$: By the base case of Definition 10, we know that $\mathcal{F}_\alpha(P, \nu, P') = 1 = \mathcal{F}_\alpha(P, \nu, P'')$ for any ν. Moreover, if there exists at least a $\hat{P} \in \mathcal{B}(P)$ such that $r_\alpha(\hat{P}) > 0$, then by Definition 3 we know that for any population function $\hat{\nu}$ assigning a positive population to $\hat{P}$ we have $r_\alpha(P, \hat{\nu}) > 0$. For these cases we can thus apply Lemma 3, ensuring that we can find a $\bar{\nu}$ such that $r_\alpha(P, \bar{\nu}) = 1$ (and $\mathcal{F}_\alpha(P, \bar{\nu}, P') = 1 = \mathcal{F}_\alpha(P, \bar{\nu}, P'')$).
- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$: Let $\alpha \in \mathcal{A}$, and $P', P'' \in \mathcal{B}(\mathcal{M})$ such that there exist two population functions $\nu_{\alpha,P'}$ an $\nu_{\alpha,P''}$ for which $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$ and $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P''}, P'') > 0$, respectively. Two cases may arise: P' and P'' both belong to the same sub-model, or one belongs to $\mathcal{M}_1$ and the other one to $\mathcal{M}_2$. We first analyse the former case and assume, without loss of generality, that $P', P'' \in \mathcal{B}(\mathcal{M}_1)$.

- $\alpha \in L$: We now prove that for $\alpha \in L$ we always have the second result, i.e. that there exists a population function $\bar{\nu}$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P'')$ and $r_\alpha(\mathcal{M}, \bar{\nu}) = 1$. By Definition 10, for $\hat{P} \in \{P', P''\}$ we have:

$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu_{\alpha,\hat{P}}, \hat{P}) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,\hat{P}}, \hat{P}) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu_{\alpha,\hat{P}})}{r_\alpha(\mathcal{M}_1, \nu_{\alpha,\hat{P}})} \ .$$

The assumption $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,\hat{P}}, \hat{P}) > 0$ assures that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,\hat{P}}, \hat{P}) > 0$, $r_\alpha(\mathcal{M}_1, \nu_{\alpha,\hat{P}}) > 0$, $r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu_{\alpha,\hat{P}}) > 0$, and thus also $r_\alpha(\mathcal{M}_2, \nu_{\alpha,\hat{P}}) > 0$. Note that, this allows us to apply the I.H. on $\mathcal{M}_1$, and, at the same time, Lemma 3 to both $\mathcal{M}_1$ and $\mathcal{M}_2$.

By I.H. on $\mathcal{M}_1$ we know that two cases might happen: the first case is that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P'')$ for any ν on $\mathcal{B}(\mathcal{M}_1)$. When this holds, by exploiting Lemma 3 for both $\mathcal{M}_1$ and $\mathcal{M}_2$ we obtain that there exist two functions ν_1, ν_2 such that $r_\alpha(\mathcal{M}_1, \nu_1) = 1$, and $r_\alpha(\mathcal{M}_2, \nu_2) = 1$. Thus, we have that $r_\alpha(\mathcal{M}, (\nu_1, \nu_2)) = 1$ (and $\mathcal{F}_\alpha(\mathcal{M}, (\nu_1, \nu_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\nu_1, \nu_2), P'')$).

The second case arising from the I.H. is that there exists a $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P'')$ and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$. By exploiting Lemma 3 for $\mathcal{M}_2$, we can choose a ν_2 such that $r_\alpha(\mathcal{M}_2, \nu_2) = 1$. Thus, we have that $r_\alpha(\mathcal{M}, (\bar{\nu}_1, \nu_2)) = 1$ (and $\mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \nu_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \nu_2), P'')$).

- $\alpha \notin L$: By Definition 10, for $\hat{P} \in \{P', P''\}$ we have:

$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu_{\alpha,\hat{P}}, \hat{P}) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,\hat{P}}, \hat{P}) \ .$$

The assumption $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,\hat{P}}, \hat{P}) > 0$ assures that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,\hat{P}}, \hat{P}) > 0$. By I.H. on $\mathcal{M}_1$ we know that two cases might happen: the first case is that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P'')$ for any ν_1 on $\mathcal{B}(\mathcal{M}_1)$. Therefore, we have that $\mathcal{F}_\alpha(\mathcal{M}, (\nu_1, \nu_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\nu_1, \nu_2), P'')$ for any ν_1 on $\mathcal{B}(\mathcal{M}_1)$ and ν_2 on $\mathcal{B}(\mathcal{M}_2)$.

The second case arising from the I.H. is that there exists a $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P'')$ and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$. Now, either the result of Lemma 3 can be applied for

$\mathcal{M}_2$, or not. In the former case, we can choose a $\hat{\nu}_2$ such that $r_\alpha(\mathcal{M}_2, \hat{\nu}_2) = 0$. In the latter, instead, we have that $r_\alpha(\mathcal{M}_2, \nu_2) = 0$ for any ν_2. Therefore, following Definition 3, we have that $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\bar{\nu}_1, \hat{\nu}_2)) = r_\alpha(\mathcal{M}_1, \bar{\nu}_1) + r_\alpha(\mathcal{M}_2, \hat{\nu}_2) = 1$ (and $\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\bar{\nu}_1, \hat{\nu}_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\bar{\nu}_1, \hat{\nu}_2), P''))$.

As previously said, we might have the case in which either both P' and P'' belong to the same sub-model, or the case when one belongs to $\mathcal{M}_1$ and the other one to $\mathcal{M}_2$. The first case having being proved, we now focus on the second one. We assume, without loss of generality, that $P' \in \mathcal{B}(\mathcal{M}_1)$ and $P'' \in \mathcal{B}(\mathcal{M}_2)$. Noteworthy, this case is addressed resorting to Lemma 3 and Lemma 4 only.

- $\alpha \in L$: We now prove that for $\alpha \in L$ we always have the second result, i.e.: there exists a population function $\bar{\nu}$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P'')$ and $r_\alpha(\mathcal{M}, \bar{\nu}) = 1$. By Definition 10 we have:

$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'}, P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'})}{r_\alpha(\mathcal{M}_1, \nu_{\alpha,P'})} ,$$
$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P''}, P'') = \mathcal{F}_\alpha(\mathcal{M}_2, \nu_{\alpha,P''}, P'') \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P''})}{r_\alpha(\mathcal{M}_2, \nu_{\alpha,P''})} .$$

The assumptions $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$ and $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P''}, P'') > 0$ assure that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') > 0$, $r_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}) > 0$, and $r_\alpha(\mathcal{M}_2, \nu_{\alpha,P'}) > 0$, as well as $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_{\alpha,P''}, P'') > 0$, $r_\alpha(\mathcal{M}_1, \nu_{\alpha,P''}) > 0$, and $r_\alpha(\mathcal{M}_2, \nu_{\alpha,P''}) > 0$. Note that this allows us to apply Lemma 3 and Lemma 4 to both $\mathcal{M}_1$ and $\mathcal{M}_2$.

We now apply Lemma 4 to $\mathcal{M}_1$ and $\mathcal{M}_2$, fixing the value of K to 1. We know that four cases might happen:

1. $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') = 1$ for any ν_1. $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_2, P'') = 1$ for any ν_2.
2. $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') = 1$ for any ν_1. There exists a $\bar{\nu}_2$ such that $\mathcal{F}_\alpha(\mathcal{M}_2, \bar{\nu}_2, P'') = 1$ and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = 1$.
3. There exists a $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P') = 1$ and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$. $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_2, P'') = 1$ for any ν_2.
4. There exists a $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P') = 1$ and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$. There exists a $\bar{\nu}_2$ such that $\mathcal{F}_\alpha(\mathcal{M}_2, \bar{\nu}_2, P'') = 1$ and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = 1$.

Case 1 follows by exploiting Lemma 3 on $\mathcal{M}_1$ and $\mathcal{M}_2$, which guarantees the existence of $\bar{\nu}_1$ and $\bar{\nu}_2$ such that $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$, and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = 1$, obtaining the claim, i.e. $\mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P') = \mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P'') = 1$ and $r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\bar{\nu}_1, \bar{\nu}_2)) = 1$, regardless of the function $\mathcal{H}$. We point out that the proof will not go through if the original statement were to be replaced with $r_\alpha(\mathcal{M}, \bar{\nu}) = K \neq 1$, unless one would consider only the interaction function $\mathcal{H} = \min$.

Case 2 follows by exploiting Lemma 3 on $\mathcal{M}_1$, which guarantees the existence of $\bar{\nu}_1$ such that $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$. Thus, $r_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2)) = 1$, and $\mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P'')$.

Case 3 is symmetric to case 2.

Case 4 implies that $r_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2)) = 1$, and $\mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P'')$.

- $\alpha \notin L$: By Definition 10 we have:

$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P'}, P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') ,$$
$$\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu_{\alpha,P''}, P'') = \mathcal{F}_\alpha(\mathcal{M}_2, \nu_{\alpha,P''}, P'') .$$

The assumptions $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P'}, P') > 0$ and $\mathcal{F}_\alpha(\mathcal{M}, \nu_{\alpha,P''}, P'') > 0$ assure that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{\alpha,P'}, P') > 0$, and $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_{\alpha,P''}, P'') > 0$. Note that this allows us to apply Lemma 4 to both $\mathcal{M}_1$ and $\mathcal{M}_2$.

As in the case $\alpha \in L$, we now apply Lemma 4 to $\mathcal{M}_1$ and $\mathcal{M}_2$ fixing $K = 1$. This again yields four cases. However, in order to prove the fourth one, we now exploit Lemma 4 by choosing the value $K = 1/2$.

1. $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') = 1$ for any ν_1. $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_2, P'') = 1$ for any ν_2.
2. $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P') = 1$ for any ν_1. There exists a $\bar{\nu}_2$ such that $\mathcal{F}_\alpha(\mathcal{M}_2, \bar{\nu}_2, P'') = 1$ and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = 1$.
3. There exists a $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P') = 1$ and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1$. $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_2, P'') = 1$ for any ν_2.
4. There exists a $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P') = 1$ and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = 1/2$. There exists a $\bar{\nu}_2$ such that $\mathcal{F}_\alpha(\mathcal{M}_2, \bar{\nu}_2, P'') = 1$ and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = 1/2$.

Case 1 implies that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P') = \mathcal{F}_\alpha(\mathcal{M}, \nu, P'') = 1$ for any ν.

As regards case 2 we have that, either the result of Lemma 3 can be applied for $\mathcal{M}_1$, or not. In the former case, we can choose a $\hat{\nu}_1$ such that $r_\alpha(\mathcal{M}_1, \hat{\nu}_1) = 0$. In the latter, instead, we have that $r_\alpha(\mathcal{M}_1, \nu_1) = 0$ for any ν_1. Therefore, from Definition 3, we have that $r_\alpha(\mathcal{M}, (\hat{\nu}_1, \bar{\nu}_2)) = r_\alpha(\mathcal{M}_1, \hat{\nu}_1) + r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = 1$ (and $\mathcal{F}_\alpha(\mathcal{M}, (\hat{\nu}_1, \bar{\nu}_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\hat{\nu}_1, \bar{\nu}_2), P'')$).

Case 3 is symmetric to case 2.

Case 4 implies that $r_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2)) = 1$, and $\mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P') = 1 = \mathcal{F}_\alpha(\mathcal{M}, (\bar{\nu}_1, \bar{\nu}_2), P'')$.

□

The next lemma says that the model impedes a local state from performing a certain dependent action if the distribution of the population within the model is such that the whole model does not perform that action.

Lemma 6. *Let $\mathcal{M}$ be a FEPA model. Let $P \in \mathcal{B}(\mathcal{M})$, and $\alpha \in \mathcal{D}(P, \mathcal{M})$. For any ν such that $r_\alpha(\mathcal{M}, \nu) = 0$, we have that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = 0$.*

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: By Definition 8, we have $\mathcal{D}(P', \mathcal{M}) = \emptyset$ for any $P' \in \mathcal{B}(P)$, thus this case is vacuously true.
- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: We assume, without loss of generality, that $P \in \mathcal{B}(\mathcal{M}_1)$. Let $\alpha \in \mathcal{D}(P, \mathcal{M})$, we distinguish among two cases: $\alpha \in L$, $\alpha \notin L$.

- $\alpha \in L$: The assumption $r_\alpha(\mathcal{M}, \nu) = 0$, and Definition 3, imply that $r_\alpha(\mathcal{M}_1, \nu) = 0$ or $r_\alpha(\mathcal{M}_2, \nu) = 0$. Moreover, Definition 10 tells us that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)}$. Given that $r_\alpha(\mathcal{M}_1, \nu) = 0$ or $r_\alpha(\mathcal{M}_2, \nu) = 0$, then we also have $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = 0$, implying $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = 0$.

- $\alpha \notin L$: The assumption $r_\alpha(\mathcal{M}, \nu) = 0$, and Definition 3, assure that $r_\alpha(\mathcal{M}_1, \nu) = 0$ and $r_\alpha(\mathcal{M}_2, \nu) = 0$. The assumptions $\alpha \in \mathcal{D}(P, \mathcal{M})$ and $\alpha \notin L$ imply that $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$, thus we can apply the I.H to $\mathcal{M}_1$. Definition 10 tells us that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$. We therefore obtain $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = 0$.

□

The last lemma of this appendix is similar in nature to Lemma 4. However, focusing on dependent actions only, it grants more freedom in the tuning of the model influence. Specifically, it says that given a model $\mathcal{M}$ and a local state $P \in \mathcal{B}(\mathcal{M})$, for any $K > 0$ and for any $\varepsilon \in (0,1)$ it is always possible to tune the population function to simultaneously make the model influence upon the rate at which P executes a certain dependent action α equal to ε, and to set to K the α-apparent rate of the model $\mathcal{M}$.

Lemma 7. *Let $\mathcal{M}$ be a well-posed FEPA model, $P \in \mathcal{B}(\mathcal{M})$ and $\alpha \in \mathcal{D}(P, \mathcal{M})$. For any $K > 0$ and any $\varepsilon \in (0,1)$ there exists a population function $\nu_{K,\varepsilon}$ such that*

$$\mathcal{F}_\alpha(\mathcal{M}, \nu_{K,\varepsilon}, P) = \varepsilon \qquad \textit{and} \qquad r_\alpha(\mathcal{M}, \nu_{K,\varepsilon}) = K\ .$$

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: By Definition 8, we have $\mathcal{D}(P', \mathcal{M}) = \emptyset$ for any $P' \in \mathcal{B}(P)$, thus this case is vacuously true.
- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_{L} \mathcal{M}_2$: We assume, without loss of generality, that $P \in \mathcal{B}(\mathcal{M}_1)$. Let $\alpha \in \mathcal{D}(P, \mathcal{M})$, we distinguish among two cases: $\alpha \in L$, $\alpha \notin L$.

\- $\alpha \in L$: From Definition 10 we have $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_{L} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)}$.

Within the case $\alpha \in L$, we further distinguish between two cases: $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$ and $\alpha \notin \mathcal{D}(P, \mathcal{M}_1)$.

i) $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$: Making use of Proposition 3 (which assures that $\mathcal{M}_1$ is also well-posed) together with the assumption $P \in \mathcal{B}(\mathcal{M}_1)$, we can use the I.H. on $\mathcal{M}_1$ to infer that for any K_1, ε_1 there exists ν_{K_1,ε_1} such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{K_1,\varepsilon_1}, P) = \varepsilon_1$ and $r_\alpha(\mathcal{M}_1, \nu_{K_1,\varepsilon_1}) = K_1$. At the same time, the well-posedness assumption on $\mathcal{M}$ assures the existence of a population function ν_2 on $\mathcal{M}_2$ such that $r_\alpha(\mathcal{M}_2, \nu_2) > 0$, allowing us to apply Lemma 3 to $\mathcal{M}_2$ which assures, for any arbitrary non-negative value K_2, the existence of a population function $\bar{\nu}_2$ such that $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = K_2$. Thus, for any K, ε we can choose K_1, ε_1, K_2 such that: if $\mathcal{H} = \min$, $\varepsilon_1 \frac{\min(K_1,K_2)}{K_1} = \varepsilon$ and $\min(K_1, K_2) = K$. In case $\mathcal{H} = \cdot$, $\varepsilon_1 K_2 = \varepsilon$ and $K_1 K_2 = K$. The corresponding population function $\nu_{K,\varepsilon} = (\nu_{K_1,\varepsilon_1}, \bar{\nu}_2)$ satisfies the claim.

ii) $\alpha \notin \mathcal{D}(P, \mathcal{M}_1)$: Exploiting Proposition 1, we have that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = 1$ for any population function ν on $\mathcal{M}_1$. Furthermore, the well-posedness assumption on $\mathcal{M}$ allows us to apply Lemma 3 to $\mathcal{M}_1$ and $\mathcal{M}_2$ which assures, for any arbitrary non-negative value K_1 (resp. K_2), the existence of a population function $\bar{\nu}_1$ (resp. $\bar{\nu}_2$) such that $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = K_1$ (resp. $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = K_2$). Thus, for any K, ε, we can choose K_1 and K_2 in Lemma 3 such that: if $\mathcal{H} = \min$, $\frac{\min(K_1,K_2)}{K_1} = \varepsilon$ and $\min(K_1, K_2) = K$. In case $\mathcal{H} = \cdot$, $K_2 = \varepsilon$ and $K_1 K_2 = K$. The corresponding population function $\nu_{K,\varepsilon} = (\bar{\nu}_1, \bar{\nu}_2)$ satisfies the claim.

\- $\alpha \notin L$: From Definition 10 we have $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$. The assumptions $\alpha \in \mathcal{D}(P, \mathcal{M})$ and $\alpha \notin L$ imply that $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$ and we can apply the I.H on $\mathcal{M}_1$. This assures that for any K_1, ε_1 there exists ν_{K_1,ε_1} such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{K_1,\varepsilon_1}, P) = \varepsilon_1$ and $r_\alpha(\mathcal{M}_1, \nu_{K_1,\varepsilon_1}) = K_1$. Due to the fact that $\alpha \notin L$ we know that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$ and, in particular, its value does not depend on the population function assigned to $\mathcal{M}_2$. Furthermore, from Definition 3, by choosing $\hat{\nu}_2$ such that $r_\alpha(\mathcal{M}_2, \hat{\nu}_2) = 0$, we know that $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}_1, \nu_1)$. Thus, for any K, ε, we can choose $K_1 = K$ and $\varepsilon_1 = \varepsilon$ and obtain that the function $(\nu_{K_1,\varepsilon_1}, \hat{\nu}_2)$ proves the claim.

□

B Results relaring DOL to ODE lumpability (Proof of Theorem 1)

Definition 17 ($\mathcal{P}$-block redistributed population function)**.** *Let $\mathcal{M}$ be a FEPA model, and $\mathcal{P}$ a partition of $\mathcal{B}(\mathcal{M})$. Let $M_{\mathcal{P}}$ denote the aggregation matrix induced by $\mathcal{P}$ on $\mathcal{M}$, and $\overline{M}_{\mathcal{P}}$ denote a generalised right inverse of $M_{\mathcal{P}}$. For any population function ν for $\mathcal{M}$ we define the $\mathcal{P}$-*block redistributed population function *$[\nu]^{\mathcal{P}}$ for $\mathcal{M}$ as*

$$[\nu]^{\mathcal{P}} = \overline{M}_{\mathcal{P}} M_{\mathcal{P}} \nu \ .$$

The condition $M_{\mathcal{P}}\overline{M}_{\mathcal{P}} = \mathbb{I}_{\mathcal{P}}$ implies that the generalised right inverse $\overline{M}_{\mathcal{P}}$ is not unique and can be parametrised by $|\mathcal{B}(\mathcal{M})|$ values. Thus, each component of the $\mathcal{P}$-block redistributed population function $[\nu]^{\mathcal{P}}$ is given by

$$[\nu]^{\mathcal{P}}{}_P = a_P \hat{\nu}_S \ ,$$

where a_P, for each $P \in \mathcal{B}(\mathcal{M})$, is one of such parameters for which it it holds that $\sum_{P\in S} a_P = 1$ for all $S \in \mathcal{P}$. Furthermore, $[\nu]^{\mathcal{P}}$ satisfies $\sum_{P\in S} [\nu]^{\mathcal{P}}{}_P = \hat{\nu}_S$.

We need two lemmata before proving the theorem.

Lemma 8. *Let $\mathcal{M}$ be an $\mathcal{A}$-coherent FEPA model, let $\hat{\mathcal{A}}$ be a set of actions, and $\mathcal{P}$ be a partition of $\mathcal{B}(\mathcal{M})$ such that for any $S \in \mathcal{P}$ and any $P, Q \in S$ it holds:*

i) P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$,

ii) $r_\beta(P) = r_\beta(Q)$, for any $\beta \in \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$.

Then, for any sub-model $\mathcal{M}'$ of $\mathcal{M}$, we have that for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}'), \mathcal{M}')$, for any ν,

$$r_\alpha(\mathcal{M}', \nu) = r_\alpha(\mathcal{M}', [\nu]^{\mathcal{P}|_{\mathcal{M}'}}) \ .$$

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: This case vacuously holds, as P does not have any sub-models except itself, and $\mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(P), P) = \emptyset$.

- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: We now focus on $\mathcal{M}_1$, but the same arguments apply to $\mathcal{M}_2$. In order to prove this case we define $\mathcal{P}|_{\mathcal{M}_1}$, i.e. the partition of $\mathcal{B}(\mathcal{M}_1)$ obtained by restricting $\mathcal{P}$ to $\mathcal{M}_1$. This partition satisfies the assumptions of the lemma, as any block of $\mathcal{P}|_{\mathcal{M}_1}$ is contained in a block of $\mathcal{P}$, and by Proposition 7 we know that the elements of each block in $\mathcal{P}|_{\mathcal{M}_1}$ are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}_1$. Furthermore, from Definition 9 it directly follows that if $\mathcal{M}$ is $\mathcal{A}$-coherent, then $\mathcal{M}_1$ is $\mathcal{A}$-coherent as well. We can thus apply the I.H. to $\mathcal{M}_1$, having that for any sub-model $\mathcal{M}_1'$ of $\mathcal{M}_1$, for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1'), \mathcal{M}_1')$, $r_\alpha(\mathcal{M}_1', \nu) = r_\alpha(\mathcal{M}_1', [\nu]^{\mathcal{P}|_{\mathcal{M}_1}|_{\mathcal{M}_1'}})$, for any ν. Given that $\mathcal{M}_1$ is a sub-model of itself, then for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$, $r_\alpha(\mathcal{M}_1, \nu) = r_\alpha(\mathcal{M}_1, [\nu]^{\mathcal{P}|_{\mathcal{M}_1}|_{\mathcal{M}_1}})$, for any ν. Note that $[\nu]^{\mathcal{P}|_{\mathcal{M}_1}|_{\mathcal{M}_1}} = [\nu]^{\mathcal{P}|_{\mathcal{M}_1}}$, for any ν, and thus $r_\alpha(\mathcal{M}_1, [\nu]^{\mathcal{P}|_{\mathcal{M}_1}|_{\mathcal{M}_1}}) = r_\alpha(\mathcal{M}_1, [\nu]^{\mathcal{P}|_{\mathcal{M}_1}})$. As above, similar arguments apply to $\mathcal{M}_2$ as well, and thus we have considered any sub-model of $\mathcal{M}$, except $\mathcal{M}$ itself.

  It remains to prove that for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}), \mathcal{M})$, $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}|_{\mathcal{M}}})$, where, clearly, $r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}|_{\mathcal{M}}}) = r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}})$. Furthermore, note that for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}), \mathcal{M})$ we have $\alpha \in \hat{\mathcal{A}}$ and $\alpha \in D(\mathcal{B}(\mathcal{M}), \mathcal{M})$.

  We call spurious the partition blocks of $\mathcal{P}$ whose elements divide among $\mathcal{M}_1$ and $\mathcal{M}_2$, and define the set of spurious blocks of $\mathcal{P}$ for $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$ as $\mathbb{S}(\mathcal{P}, \mathcal{M}) = \{\tilde{S} \in \mathcal{P} \mid \tilde{S} \cap \mathcal{B}(\mathcal{M}_1) \neq \emptyset \wedge \tilde{S} \cap \mathcal{B}(\mathcal{M}_2) \neq \emptyset\}$. We shall indicate with K the union of all the local states of the blocks in $\mathbb{S}(\mathcal{P}, \mathcal{M})$. By Definition 13 we know that $\mathcal{CD}^{\hat{\mathcal{A}}}(K, \mathcal{M}) = \emptyset$. Given that $\alpha \in \hat{\mathcal{A}}$, we can conclude that $\alpha \notin \mathcal{CD}(K, \mathcal{M})$, allowing us to apply Lemma 1 to $\mathcal{M}$ for the set of local states K, obtaining

$r_\alpha(\mathcal{M},\nu) = r_\alpha(\mathcal{M},\nu^K) + \sum_{P\in K} \nu_P r_\alpha(P)$, where ν^K is defined as $\nu^K_P = \nu_P$ if $P \notin K$ and $\nu^K_P = 0$ if $P \in K$.

For all partition blocks in $S \in (\mathcal{P} \setminus \mathbb{S}(\mathcal{P},\mathcal{M}))$, we have instead that either $S \subseteq \mathcal{B}(\mathcal{M}_1)$, and thus $S \in \mathcal{P}|_{\mathcal{M}_1}$, or $S \subseteq \mathcal{B}(\mathcal{M}_2)$, and thus $S \in \mathcal{P}|_{\mathcal{M}_2}$.

By resorting to standard set theory we note that

$$\mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}),\mathcal{M}) = \bigcup_{P\in\mathcal{B}(\mathcal{M}_1)} (\mathcal{A}(P) \cap \hat{\mathcal{A}} \cap L) \ \cup \ \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1) \ \cup \bigcup_{P\in\mathcal{B}(\mathcal{M}_2)} (\mathcal{A}(P) \cap \hat{\mathcal{A}} \cap L) \ \cup \ \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_2),\mathcal{M}_2) \ .$$

From the I.H. we know that

- for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1)$, $r_\alpha(\mathcal{M}_1,\nu) = r_\alpha(\mathcal{M}_1,[\nu]^{\mathcal{P}|_{\mathcal{M}_1}})$,
- for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_2),\mathcal{M}_2)$, $r_\alpha(\mathcal{M}_2,\nu) = r_\alpha(\mathcal{M}_2,[\nu]^{\mathcal{P}|_{\mathcal{M}_2}})$.

By selecting an $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}),\mathcal{M})$, two cases have to be considered: $\alpha \notin L$ and $\alpha \in L$:

- $\alpha \notin L$: For those α we have $r_\alpha(\mathcal{M},\nu) = r_\alpha(\mathcal{M},\nu^K) + \sum_{P\in K} \nu_P r_\alpha(P) = r_\alpha(\mathcal{M}_1,\nu^K) + r_\alpha(\mathcal{M}_2,\nu^K) + \sum_{P\in K} \nu_P r_\alpha(P)$. First of all, we note that for any $P \in K$ we have $r_\alpha(P) = 0$, and thus $\sum_{P\in K} \nu_P r_\alpha(P) = 0$. In fact, given that $\mathcal{CD}^{\hat{\mathcal{A}}}(K,\mathcal{M}) = \emptyset$ and $\alpha \in \hat{\mathcal{A}}$, we have $\alpha \notin \bigcup_{P\in K} \mathcal{A}(P) \cap \mathcal{D}(P,\mathcal{M})$. For every $P \in K$ we may have $\alpha \notin \mathcal{A}(P)$, and thus $r_\alpha(P) = 0$, or $\alpha \notin \mathcal{D}(P,\mathcal{M})$. In the latter case, given that $\alpha \in \mathcal{D}(\mathcal{B}(\mathcal{M}),\mathcal{M})$, by the $\mathcal{A}$-coherence property of $\mathcal{M}$ we again have $r_\alpha(P) = 0$.

  We remark that if $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_i),\mathcal{M}_i)$, for $i \in \{1,2\}$, we can exploit the I.H. on $\mathcal{M}_i$ obtaining $r_\alpha(\mathcal{M}_i,\nu^K) = r_\alpha(\mathcal{M}_i,[\nu^K]^{\mathcal{P}|_{\mathcal{M}_i}})$. Given that ν^K assigns population 0 to all the elements of the spurious blocks in $\mathbb{S}(\mathcal{P},\mathcal{M})$, we have that $r_\alpha(\mathcal{M}_i,[\nu^K]^{\mathcal{P}|_{\mathcal{M}_i}})$ depends only on the blocks of $\mathcal{P}$ fully contained in $\mathcal{B}(\mathcal{M}_i)$, i.e. only on the blocks of $\mathcal{P}|_{\mathcal{M}_i}$ that also belong to $\mathcal{P}$. We can thus write $r_\alpha(\mathcal{M}_i,[\nu^K]^{\mathcal{P}|_{\mathcal{M}_i}}) = r_\alpha(\mathcal{M}_i,[\nu^K]^{\mathcal{P}})$.

  Given that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}),\mathcal{M})$ and $\alpha \notin L$, by resorting to standard set theory we obtain $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1) \cup \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_2),\mathcal{M}_2)$. In case α belongs to both $\mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1)$ and $\mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_2),\mathcal{M}_2)$, then $r_\alpha(\mathcal{M},\nu^K) = r_\alpha(\mathcal{M}_1,[\nu^K]^{\mathcal{P}|_{\mathcal{M}_1}}) + r_\alpha(\mathcal{M}_2,[\nu^K]^{\mathcal{P}|_{\mathcal{M}_2}}) = r_\alpha(\mathcal{M}_1,[\nu^K]^{\mathcal{P}}) + r_\alpha(\mathcal{M}_2,[\nu^K]^{\mathcal{P}})$. If instead $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1)$ and $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_2),\mathcal{M}_2)$ (or vice versa), we can apply Lemma 1 to $\mathcal{M}_1$ for the local states $\mathcal{B}(\mathcal{M}_1)$, obtaining $r_\alpha(\mathcal{M}_1,\nu^K) = \sum_{P\in\mathcal{B}(\mathcal{M}_1)} \nu^K_P r_\alpha(P)$. Note that Lemma 1 can be applied due to the fact that $\alpha \in \hat{\mathcal{A}}$ and $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1)$ implies $\alpha \notin \mathcal{CD}(\mathcal{B}(\mathcal{M}_1),\mathcal{M}_1))$. Similarly to how above discussed, the last summation depends only on the blocks of $\mathcal{P}$ fully contained in $\mathcal{B}(\mathcal{M}_1)$, i.e. only on the blocks of $\mathcal{P}|_{\mathcal{M}_1}$ that also belong to $\mathcal{P}$. We can thus rewrite the last summation as

$$\sum_{\substack{S\in\mathcal{P} \\ S\subseteq\mathcal{B}(\mathcal{M}_1)}} \sum_{P\in S} \nu^K_P r_\alpha(P) \ .$$

  For any of the considered S we now can have two cases: $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(S,\mathcal{M})$ or $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(S,\mathcal{M})$. In the former case, by the assumptions of the lemma we have $r_\alpha(P) = r_\alpha(Q)$, for any $P,Q \in S$. In the latter case, similarly to how above discussed, due to the $\mathcal{A}$-coherence of $\mathcal{M}$ we have $r_\alpha(P) = 0$ for any $P \in S$. In fact, $\alpha \in \mathcal{D}(\mathcal{B}(\mathcal{M}),\mathcal{M})$, but $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(S,\mathcal{M})$ and $\alpha \in \hat{\mathcal{A}}$, and thus $\alpha \notin \mathcal{CD}(S,\mathcal{M})$. Now, for any $P \in S$, we have $\mathcal{A}(P) = 0$ (and thus $r_\alpha(P) = 0$), or $\alpha \notin \mathcal{D}(P,\mathcal{M})$, which by $\mathcal{A}$-coherence implies $r_\alpha(P) = 0$. In both cases we can write

$$r_\alpha(\mathcal{M}_1, \nu^K) = \sum_{\substack{S\in\mathcal{P}\\ S\subseteq\mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P\in S} \nu^K_P = \sum_{\substack{S\in\mathcal{P}\\ S\subseteq\mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P\in S} [\nu^K]^{\mathcal{P}}{}_P \ ,$$

where $r_\alpha(S) \triangleq r_\alpha(P)$, for any $P \in S$.

We thus have

$$r_\alpha(\mathcal{M}, \nu^K) = r_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}}) + r_\alpha(\mathcal{M}_2, [\nu^K]^{\mathcal{P}} = r_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}}) \ .$$

We now recall that, in this particular case, we have $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}, \nu^K)$. This holds for any ν, and thus also for $[\nu]^{\mathcal{P}}$, yielding $r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}) = r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}K})$. Finally, given that $r_\alpha(\mathcal{M}, \nu^K) = r_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}})$, the original claim $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}})$ follows by the fact that $[\nu^K]^{\mathcal{P}} = [\nu]^{\mathcal{P}K}$, for any ν. In fact, in the left-hand side of the equality we first set to zero the population of the local states of the spurious blocks (i.e. the local states K), and then, for each block, we redistribute the cumulative population of the block among its local states. Conversely, in the right-hand side we first redistribute the population in each block, and then we set to zero the population of the local states of the spurious blocks. Clearly, redistributing a zero population within a block is equal to redistributing any population within a block, and then set to zero the population of all its local states.

- $\alpha \in L$: For those α we have $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu^K) + \sum_{P\in K} \nu_P r_\alpha(P)$, where

$$r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu^K) \triangleq \begin{cases} \min(r_\alpha(\mathcal{M}_1, \nu^K), r_\alpha(\mathcal{M}_2, \nu^K)) \ , & \text{if } \mathcal{H} = \min \ , \\ \\ r_\alpha(\mathcal{M}_1, \nu^K) \cdot r_\alpha(\mathcal{M}_2, \nu^K) \ , & \text{if } \mathcal{H} = \cdot \ . \end{cases}$$

First of all, we note that for any $P \in K$ we have $r_\alpha(P) = 0$, and thus $\sum_{P\in K} \nu^K_P r_\alpha(P) = 0$, as we have $\alpha \in \mathcal{D}(K, \mathcal{M})$ and $\alpha \notin \mathcal{CD}(K, \mathcal{M})$. This is directly implied by the fact that $\alpha \in L$ (and thus $\alpha \in \mathcal{D}(K, \mathcal{M})$), and by the fact that $\alpha \in \hat{\mathcal{A}}$ together with $\mathcal{CD}^{\hat{\mathcal{A}}}(K, \mathcal{M}) = \emptyset$ (and thus $\alpha \notin \mathcal{CD}(K, \mathcal{M})$).

Let us now focus on $\mathcal{M}_1$. We can have either $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$ or $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$. In the case $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$, by I.H., we have $r_\alpha(\mathcal{M}_1, \nu^K) = r_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}|_{\mathcal{M}_1}})$. Similarly to how discussed in the $\alpha \notin L$ case, we have that $r_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}|_{\mathcal{M}_1}})$ depends only on the blocks of $\mathcal{P}$ fully contained in $\mathcal{B}(\mathcal{M}_1)$, i.e. only on the blocks of $\mathcal{P}|_{\mathcal{M}_1}$ that also belong to $\mathcal{P}$. We can thus write $r_\alpha(\mathcal{M}_1, \nu^K) = r_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}})$. The same holds for $\mathcal{M}_2$. In the case $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$ instead, we can apply Lemma 1 to $\mathcal{M}_1$ for the local states $\mathcal{B}(\mathcal{M}_1)$, obtaining $r_\alpha(\mathcal{M}_1, \nu^K) = \sum_{P\in\mathcal{B}(\mathcal{M}_1)} \nu^K_P r_\alpha(P)$. Note that Lemma 1 can be applied due to the fact that $\alpha \in \hat{\mathcal{A}}$ and $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$ implies $\alpha \notin \mathcal{CD}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1))$. Similarly to how above discussed, the last summation depends only on the blocks of $\mathcal{P}$ fully contained in $\mathcal{B}(\mathcal{M}_1)$, i.e. only on the blocks of $\mathcal{P}|_{\mathcal{M}_1}$ that also belong to $\mathcal{P}$. We can thus rewrite the last summation as

$$\sum_{\substack{S\in\mathcal{P}\\ S\subseteq\mathcal{B}(\mathcal{M}_1)}} \sum_{P\in S} \nu^K_P r_\alpha(P) \ .$$

For any of the considered S we now can have two cases: $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$ or $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$. In the former case, by the assumptions of the lemma we have $r_\alpha(P) = r_\alpha(Q)$, for any $P, Q \in S$. In the latter case, due the the $\mathcal{A}$-coherence of $\mathcal{M}$, given that $\alpha \notin \mathcal{CD}(S, \mathcal{M})$ (as $\alpha \in \hat{\mathcal{A}}$ and

$\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$), but α is a synchronized action in $\mathcal{M}$ (as $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}), \mathcal{M})$), we must have that $r_\alpha(P) = 0$ for any $P \in S$. Therefore, in both cases, we can write

$$r_\alpha(\mathcal{M}_1, \nu^K) = \sum_{\substack{S \in \mathcal{P} \\ S \subseteq \mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P \in S} \nu^K_P = \sum_{\substack{S \in \mathcal{P} \\ S \subseteq \mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P \in S} [\nu^K]^{\mathcal{P}}{}_P \ ,$$

where $r_\alpha(S) \triangleq r_\alpha(P)$, for any $P \in S$. We thus have $r_\alpha(\mathcal{M}, \nu^K) = r_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}})$.

As done for the $\alpha \notin L$ case, we now recall that $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}, \nu^K)$ for any ν, yielding $r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}) = r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}^K})$. Finally, given that $r_\alpha(\mathcal{M}, \nu^K) = r_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}})$, the original claim $r_\alpha(\mathcal{M}, \nu) = r_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}})$ follows by the fact that $[\nu^K]^{\mathcal{P}} = [\nu]^{\mathcal{P}^K}$, for any ν. The last equivalence can be justified by resorting to the same arguments used for the $\alpha \notin L$ case.

□

Lemma 9. *Let $\mathcal{M}$ be an $\mathcal{A}$-coherent FEPA model, $\hat{\mathcal{A}}$ a set of actions, and $\mathcal{P}$ be a partition of $\mathcal{B}(\mathcal{M})$ such that for any $S \in \mathcal{P}$ and any $P, Q \in S$ it holds:*

i) P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$,

ii) $r_\beta(P) = r_\beta(Q)$, for any $\beta \in \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$.

Then, for any $P \in \mathcal{B}(\mathcal{M})$, for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and for any ν it holds that

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}, P) \ .$$

Proof. First of all, we note that, for any $P \in \mathcal{B}(\mathcal{M})$, $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}(P, \mathcal{M}) \cap \hat{\mathcal{A}}$. We thus only have to focus on the actions in the set $\hat{\mathcal{A}}$. We prove the claim using structural induction.

- $\mathcal{M} = P$: For any $P \in \mathcal{B}(P)$ we have $\mathcal{CD}^{\hat{\mathcal{A}}}(P, P) = \emptyset$, thus the claim is vacuously true.
- $\mathcal{M} = \mathcal{M}_1 \parallel_L \mathcal{M}_2$: We fix a $P \in \mathcal{B}(\mathcal{M})$ and we assume, without loss of generality, that $P \in \mathcal{B}(\mathcal{M}_1)$. We call spurious the partition blocks of $\mathcal{P}$ whose elements divide among $\mathcal{M}_1$ and $\mathcal{M}_2$, and define the set of spurious blocks of $\mathcal{P}$ for $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$ as $\mathbb{S}(\mathcal{P}, \mathcal{M}) = \{\tilde{S} \in \mathcal{P} \mid \tilde{S} \cap \mathcal{B}(\mathcal{M}_1) \neq \emptyset \wedge \tilde{S} \cap \mathcal{B}(\mathcal{M}_2) \neq \emptyset\}$. We shall indicate with K the union of all the local states of the blocks in $\mathbb{S}(\mathcal{P}, \mathcal{M})$. Noteworthy, from Definition 13 we know that $\mathcal{CD}^{\hat{\mathcal{A}}}(K, \mathcal{M}) = \emptyset$. This ensures that $\alpha \notin \mathcal{CD}(K, \mathcal{M})$ for any $\alpha \in \hat{\mathcal{A}}$. Hence, we know from Lemma 2 that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu^K, P)$ for any ν, and $\alpha \in \hat{\mathcal{A}}$. We thus reduce the problem to proving that the claim holds for any ν^K, i.e., we prove that for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and for any ν^K it holds that

  $$\mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) = \mathcal{F}_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}}, P) \ .$$

  The original claim will then follow noticing that for any ν it holds that $[\nu^K]^{\mathcal{P}} = [\nu]^{\mathcal{P}^K}$. The latter equality is due to the fact that spurious blocks have zero population. Therefore, redistributing a zero population within a partition block is equal to redistributing any population function within the same block and then set its value to zero.

  We need to distinguish among two cases, $\alpha \notin L$, $\alpha \in L$.

  - $\alpha \notin L$: By Definition 10 we have

  $$\mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) = \mathcal{F}_\alpha(\mathcal{M}_1, (\nu^K)_1, P) \ .$$

  Let $\mathcal{P}|_{\mathcal{M}_1} = \{S \cap \mathcal{B}(\mathcal{M}_1) \mid S \in \mathcal{P}\}$. We remark that this partition satisfies the assumptions of the lemma. In fact, by Proposition 7, we know that $\mathcal{P}|_{\mathcal{M}_1}$ is a partition of $\mathcal{B}(\mathcal{M}_1)$ such that for any $S' \in \mathcal{P}|_{\mathcal{M}_1}$ and any $P, Q \in S'$, we have that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}_1$. Furthermore, any block of $\mathcal{P}|_{\mathcal{M}_1}$ is contained in a block of $\mathcal{P}$, assuring thus the second assumption as well. If $\alpha \notin L$ and $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$, we have $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ and we can apply the I.H. to obtain

$$\mathcal{F}_\alpha(\mathcal{M}_1, (\nu^K)_1, P) = \mathcal{F}_\alpha(\mathcal{M}_1, [(\nu^K)_1]^{\mathcal{P}|_{\mathcal{M}_1}}, P) \ .$$

As a last step to obtain the claim, we notice that for any $(\nu^K)_1$ we have $[(\nu^K)_1]^{\mathcal{P}|_{\mathcal{M}_1}} = ([\nu^K]^{\mathcal{P}})_1$. The above equality deserves a discussion. On the left-hand side we first set to zero the population function of the spurious blocks, then we project on $\mathcal{M}_1$. In oher words, we consider only local states in $\mathcal{B}(\mathcal{M}_1)$ and redistribute the total population of the blocks in $\mathcal{P}|_{\mathcal{M}_1}$ among its elements. On the right-hand side instead, we first set to zero the population function in each spurious block, we then redistribute the obtained population among the elements of the whole blocks, and we finally project on $\mathcal{M}_1$. The result of these two different operations is, however, the same. In fact, as regards the blocks which are not spurious, the operation is the same, while as regards the spurious blocks, in both cases the corresponding population is set to zero.

Hence, we obtain

$$\begin{aligned}\mathcal{F}_\alpha(\mathcal{M}_1, (\nu^K)_1, P) &= \mathcal{F}_\alpha(\mathcal{M}_1, ([\nu^K]^{\mathcal{P}})_1, P) = \mathcal{F}_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}}, P) \\ &\overset{\alpha \notin L}{=} \mathcal{F}_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}}, P) \ .\end{aligned}$$

- $\alpha \in L$: By Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, \nu^K)}{r_\alpha(\mathcal{M}_1, \nu^K)} \ .$$

As regards the fraction appearing in the above expression, we can apply Lemma 8 to its numerator, obtaining

$$r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, \nu^K) = r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}}) \ .$$

As regards the denominator of the fraction, given that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}), \mathcal{M})$ and $\alpha \in L$, we now may have two cases: either $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$, or $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$.

- $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$: In this case we can apply Lemma 8 also to the denominator of the fraction, obtaining

$$\frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, \nu^K)}{r_\alpha(\mathcal{M}_1, (\nu^K)_1)} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}})}{r_\alpha(\mathcal{M}_1, [(\nu^K)_1]^{\mathcal{P}|_{\mathcal{M}_1}})} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}})}{r_\alpha(\mathcal{M}_1, ([\nu^K]^{\mathcal{P}})_1)} \ .$$

If $\alpha \in L$ and $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ we might have that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ or $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$. In the first case, similarly to the $\alpha \notin L$ case, we can use the I.H. on $\mathcal{M}_1$, obtaining

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P) = \mathcal{F}_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}}, P) \ .$$

As regards the second scenario, instead, we have that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$. Recalling the definition of current dependent action set we have that

$$\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{A}(P) \cap \hat{\mathcal{A}} \cap \mathcal{D}(P, \mathcal{M}) = \big(\mathcal{A}(P) \cap \hat{\mathcal{A}} \cap \mathcal{D}(P, \mathcal{M}_1)\big) \cup \Big(\mathcal{A}(P) \cap \hat{\mathcal{A}} \cap L\Big) \ .$$

Which is equal to $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \cup \Big(\mathcal{A}(P) \cap \hat{\mathcal{A}} \cap L\Big)$. Thus, we can conclude that $\alpha \in \mathcal{A}(P) \cap \hat{\mathcal{A}} \cap L$ and $\alpha \notin \mathcal{A}(P) \cap \hat{\mathcal{A}} \cap \mathcal{D}(P, \mathcal{M}_1)$, from which in turn we can infer that $\alpha \notin \mathcal{D}(P, \mathcal{M}_1)$.

Thereby, Proposition 1 can be exploited to infer that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P) = 1$ for any ν^K and so also for $[\nu^K]^{\mathcal{P}}$. Consequently,

$$\mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) = \mathcal{F}_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}}, P)\frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}})}{r_\alpha(\mathcal{M}_1, [\nu^K]^{\mathcal{P}})} \overset{\alpha \in L}{=} \mathcal{F}_\alpha(\mathcal{M}, [\nu^K]^{\mathcal{P}}, P) ,$$

obtaining the claim.

- $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$: We first of all note that $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$ implies $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$. As done in the $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$, we can thus conclude that $\alpha \notin \mathcal{D}(P, \mathcal{M}_1)$. Thereby, Proposition 1 can be exploited to infer that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P) = 1$ for any ν^K and so also for $[\nu^K]^{\mathcal{P}}$. Consequently,

$$\mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu^K, P)\frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, \nu^K)}{r_\alpha(\mathcal{M}_1, \nu^K)} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, \nu^K)}{r_\alpha(\mathcal{M}_1, \nu^K)} .$$

Given that $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$, we cannot apply Lemma 8 to the denominator of the fraction, allowing us only to rewrite the fraction as

$$\frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, \nu^K)}{r_\alpha(\mathcal{M}_1, (\nu^K)_1)} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}})}{r_\alpha(\mathcal{M}_1, (\nu^K)_1)} .$$

However, given that $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(\mathcal{B}(\mathcal{M}_1), \mathcal{M}_1)$, we can apply Lemma 1 with $K = \mathcal{B}(\mathcal{M}_1)$ to the denominator, obtaining

$$r_\alpha(\mathcal{M}_1, (\nu^K)_1) = \sum_{P' \in \mathcal{B}(M_1)} r_\alpha(P')\nu^K_{P'} = \sum_{S \in \mathcal{P}|_{\mathcal{M}_1}} \sum_{P' \in S} r_\alpha(P')\nu^K_{P'} .$$

Given that ν^K assigns 0 population to the elements of the spurious blocks in $\mathbb{S}(\mathcal{P}, \mathcal{M})$, the last summation acutally depends only on the non-spurious blocks of $\mathcal{P}|_{\mathcal{M}_1}$, i.e. only on the blocks of $\mathcal{P}|_{\mathcal{M}_1}$ that also belong to $\mathcal{P}$. We can thus rewrite the last summation as

$$\sum_{\substack{S \in \mathcal{P} \\ S \subseteq \mathcal{B}(\mathcal{M}_1)}} \sum_{P' \in S} \nu^K_{P'} r_\alpha(P') .$$

For any of the considered S we now can have two cases: $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$ or $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$. In the former case, by the assumptions of the lemma we have $r_\alpha(P') = r_\alpha(P'')$, for any $P', P'' \in S$. In the latter case, due the the $\mathcal{A}$-coherence of $\mathcal{M}$, given that $\alpha \notin \mathcal{CD}(S, \mathcal{M})$ (as $\alpha \in \hat{\mathcal{A}}$ and $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(S, \mathcal{M})$), but α is a synchronized action in $\mathcal{M}$ (as $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$), we must have that $r_\alpha(P') = 0$ for any $P' \in S$. Therefore, in both cases, we can write

$$r_\alpha(\mathcal{M}_1, (\nu^K)_1) = \sum_{\substack{S \in \mathcal{P} \\ S \subseteq \mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P' \in S} \nu^K_{P'} = \sum_{\substack{S \in \mathcal{P} \\ S \subseteq \mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P' \in S} {[\nu^K]^{\mathcal{P}}}_{P'} ,$$

where $r_\alpha(S) \triangleq r_\alpha(P')$, for any $P' \in S$.
Following similar reasonings we can infer that

$$r_\alpha(\mathcal{M}_1, ([\nu^K]^{\mathcal{P}})_1) = \sum_{\substack{S \in \mathcal{P} \\ S \subseteq \mathcal{B}(\mathcal{M}_1)}} r_\alpha(S) \sum_{P' \in S} {[\nu^K]^{\mathcal{P}}}_{P'} ,$$

allowing us to conclude that

$$\mathcal{F}_\alpha(\mathcal{M}, \nu^K, P) = \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}})}{r_\alpha(\mathcal{M}_1, (\nu^K)_1)} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L \mathcal{M}_2, [\nu^K]^{\mathcal{P}})}{r_\alpha(\mathcal{M}_1, ([\nu^K]^{\mathcal{P}})_1)} ,$$

closing the case

□

We are now ready for proving the theorem.

Theorem 1. *Let $\mathcal{M}$ ba a well-posed and $\mathcal{A}$-coherent FEPA model, $\mathcal{P}$ be a DOLP of $\mathcal{B}(\mathcal{M})$. Let $M_{\mathcal{P}}$ be the aggregation matrix induced by $\mathcal{P}$ on $\mathcal{M}$, that is the $|\mathcal{P}| \times |\mathcal{B}(\mathcal{M})|$ matrix with entries 0 or 1 defined as*

$$(M_{\mathcal{P}})_{i,j} \triangleq \begin{cases} 1 & \text{if } P_j \in S_i \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

where S_i, with $i \in \{1, \ldots, |\mathcal{P}|\}$, is a block of the partition $\mathcal{P}$ and P_j, with $j \in \{1, \ldots, |\mathcal{B}(\mathcal{M})|\}$, is a local state of the model $\mathcal{M}$. Let $\hat{f} \triangleq M_{\mathcal{P}} \circ f \circ \overline{M}_{\mathcal{P}}$. Then $\hat{\nu}(t)$, solution of the ODE system

$$\dot{\hat{\nu}} = \hat{f}(\hat{\nu}), \qquad \text{with initial condition} \quad \hat{\nu}(0) = M_{\mathcal{P}}\nu_0 \ ,$$

satisfies $\hat{\nu}(t) = M_{\mathcal{P}}\nu(t)$, where $\nu(t)$ is solution of the ODE system of $\mathcal{M}$.

Proof. The proof appeals to three main points:

- A necessary and sufficient condition for the exact lumpability of the ODE system $\dot{\nu} = f(\nu)$ by the matrix $M_{\mathcal{P}}$ i.e.,

$$M_{\mathcal{P}} f(\nu) = M_{\mathcal{P}} f(\overline{M}_{\mathcal{P}} M_{\mathcal{P}} \nu) \ , \qquad \text{for all } \nu \ , \tag{5}$$

  and for any generalised right inverse $\overline{M}_{\mathcal{P}}$.

- A rewriting of the vector field f of $\mathcal{M}$ in terms of the model influence function $\mathcal{F}$.

- Lemma 9.

To verify Equation (5), recalling the definition of the aggregation matrix $M_{\mathcal{P}}$, it is enough to show that for any $S \in \mathcal{P}$ and for any ν it holds that

$$\sum_{P \in S} f_P(\nu) = \sum_{P \in S} f_P(\overline{M}_{\mathcal{P}} M_{\mathcal{P}} \nu) = \sum_{P \in S} f_P([\nu]^{\mathcal{P}}) \ .$$

In other words, we verify Equation (5) componentwise. Exploiting Definition 10 for the model influence upon the component rate, the vector field of the FEPA model $\mathcal{M}$ can be rewritten as

$$f_P(\nu) = \sum_{\alpha \in \mathcal{A}} \sum_{P' \in \mathcal{B}(\mathcal{M})} \frac{q(P', P, \alpha)}{r_\alpha(P')} \mathcal{R}_\alpha(\mathcal{M}, \nu, P') - \sum_{\alpha \in \mathcal{A}} \mathcal{R}_\alpha(\mathcal{M}, \nu, P) \ ,$$

$$\begin{aligned} &\sum_{\alpha \in \mathcal{A}} \sum_{P' \in \mathcal{B}(\mathcal{M})} \frac{q(P', P, \alpha)}{r_\alpha(P')} r_\alpha(P') \cdot \nu_{P'} \cdot \mathcal{F}_\alpha(\mathcal{M}, \nu, P') - \sum_{\alpha \in \mathcal{A}} r_\alpha(P) \cdot \nu_P \cdot \mathcal{F}_\alpha(\mathcal{M}, \nu, P) \\ &= \sum_{\alpha \in \mathcal{A}} \sum_{P' \in \mathcal{B}(\mathcal{M})} q(P', P, \alpha) \cdot \nu_{P'} \cdot \mathcal{F}_\alpha(\mathcal{M}, \nu, P') - \sum_{\alpha \in \mathcal{A}} \nu_P \left(\sum_{\tilde{S} \in \mathcal{P}} q[P, \tilde{S}, \alpha] \right) \mathcal{F}_\alpha(\mathcal{M}, \nu, P) \ . \end{aligned}$$

Summing both sides over $P \in S$ and using that $\sum_{P\in S} q(P',P,\alpha) = q[P',S,\alpha]$, as well as a decomposition of the sum over states, that is, $\sum_{P'\in\mathcal{B}(\mathcal{M})} = \sum_{S\in\mathcal{P}}\sum_{P'\in S}$, we obtain

$$
\begin{aligned}
&\sum_{P\in S} f_P(\nu)\\
&= \sum_{\alpha\in\mathcal{A}}\sum_{P'\in\mathcal{B}(\mathcal{M})} \nu_{P'}\, q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P') - \sum_{P\in S}\nu_P\left(\sum_{\tilde{S}\in\mathcal{P}}\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\\
&= \sum_{\alpha\in\mathcal{A}}\sum_{\tilde{S}\in\mathcal{P}}\sum_{P'\in\tilde{S}} \nu_{P'}\, q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P') - \sum_{P\in S}\nu_P\left(\sum_{\tilde{S}\in\mathcal{P}}\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\\
&= \sum_{\tilde{S}\in\mathcal{P}}\sum_{P'\in\tilde{S}}\left(\sum_{\alpha\in\mathcal{A}} q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P')\right)\nu_{P'} - \sum_{P\in S}\left(\sum_{\tilde{S}\in\mathcal{P}}\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\nu_P
\end{aligned}
$$

At this point we notice that the term corresponding to the chosen partition block $S \in \mathcal{P}$ in the summation $\sum_{\tilde{S}\in\mathcal{P}}$ is the same in both terms (the incoming and outgoing flux). More precisely, the first summation can be written

$$
\begin{aligned}
\sum_{\tilde{S}\in\mathcal{P}}\sum_{P'\in\tilde{S}}\left(\sum_{\alpha\in\mathcal{A}} q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P')\right)\nu_{P'} &= \underline{\sum_{P'\in S}\left(\sum_{\alpha\in\mathcal{A}} q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P')\right)\nu_{P'}} +\\
&\sum_{\tilde{S}\in\mathcal{P}/S}\sum_{P'\in\tilde{S}}\left(\sum_{\alpha\in\mathcal{A}} q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P')\right)\nu_{P'}
\end{aligned}
$$

and the second

$$
\begin{aligned}
&\sum_{P\in S}\left(\sum_{\tilde{S}\in\mathcal{P}}\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\nu_P =\\
&\sum_{P\in S}\left(\sum_{\alpha\in\mathcal{A}} q[P,S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P) + \sum_{\tilde{S}\in\mathcal{P}/S}\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\nu_P =\\
&\underline{\sum_{P\in S}\left(\sum_{\alpha\in\mathcal{A}} q[P,S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\nu_P} + \sum_{P\in S}\left(\sum_{\tilde{S}\in\mathcal{P}/S}\left(\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\right)\nu_P
\end{aligned}
$$

Therefore, we can erase the corresponding contribution obtaining

$$
\begin{aligned}
\sum_{P\in S} f_P(\nu) = &\sum_{\tilde{S}\in\mathcal{P}/S}\sum_{P'\in\tilde{S}}\left(\sum_{\alpha\in\mathcal{A}} q[P',S,\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P')\right)\nu_{P'}\\
&-\sum_{P\in S}\left(\sum_{\tilde{S}\in\mathcal{P}/S}\left(\sum_{\alpha\in\mathcal{A}} q[P,\tilde{S},\alpha]\mathcal{F}_\alpha(\mathcal{M},\nu,P)\right)\right)\nu_P\,.
\end{aligned}
\tag{6}
$$

The cancellation of these two terms has a rather intuitive explanation worth mentioning. The summation of the vector fields associated to the local states within the partition block S is made of a positive and a negative term. The positive term corresponds to the incoming fluxes within the block S coming from any other block $\tilde{S}$ of the partition, whereas the negative term corresponds to the outgoing fluxes from the block S to any other block $\tilde{S}$. When accounting for the possible contribution, the case $\tilde{S} = S$ is numbered amongst them. This case corresponds to the situation in which the block S is at the same time, source and sink. Clearly, the "self" incoming flux coincides with the "self" outgoing one.

We are left with showing that for any ν, the right-hand of (6) does not change if we replace ν with the corresponding $[\nu]^{\mathcal{P}}$.

In order to see this, we first point out that for any $S, \tilde{S} \in \mathcal{P}$ with $S \neq \tilde{S}$ and for any $P \in S$ the sum $\sum_{\alpha\in\mathcal{A}} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P)$ can be decomposed as follows

$$\sum_{\alpha\in\mathcal{A}} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) \ .$$

Indeed, due to the fact that $P \in S$ with $S \neq \tilde{S}$, only external actions bring a positive contribution to the left-hand side. The sum over external actions can be further decomposed according to the dependent action set as shown below

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \\
&\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\cap\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) + \sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\setminus\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \\
&\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\cap\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) + \sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\setminus\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha] \ ,
\end{aligned} \tag{7}$$

where in the last equality we exploited Proposition 1 which assures that for all actions $\alpha \notin \mathcal{D}(P, \mathcal{M})$, it holds that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = 1$, for any ν. Each of the two summations can be rewritten as

$$\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\cap\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \sum_{\alpha\in\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) \ ,$$

$$\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\setminus\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha] = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P,\mathcal{M})} q[P, \tilde{S}, \alpha] \ .$$

The first rewriting follows from observing that actions not in $\mathcal{A}(P)$ do not contribute to the left-hand side and by Definition 8 for the restricted current dependent action set. As far as the second equality is concerned, we first remark that $\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\setminus\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha] = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]$ as $P \notin \tilde{S}$, and thus $q[P, \tilde{S}, \alpha] = 0$ for any $\alpha \in \mathcal{A} \setminus \mathcal{A}^{\mathcal{P}}_{\text{ext}}$. Similarly, $q[P, \tilde{S}, \alpha] = 0$, for any $\alpha \in \mathcal{A} \setminus \mathcal{A}(P)$. In light of this, we have

$$\sum_{\alpha\in\mathcal{A}^{\mathcal{P}}_{\text{ext}}\setminus\mathcal{D}(P,\mathcal{M})} q[P, \tilde{S}, \alpha] = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{D}(P,\mathcal{M})\cup\mathcal{A}\setminus\mathcal{A}^{\mathcal{P}}_{\text{ext}}\cup\mathcal{A}\setminus\mathcal{A}(P)} q[P, \tilde{S}, \alpha] = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P,\mathcal{M})} q[P, \tilde{S}, \alpha] \ ,$$

where, for the last equality, we resort to standard set theory.

At this point we use the assumption that the partition considered is indeed differential ordinary lumpable. We exploit this information in two ways:

- It assures that for all $S \in \mathcal{P}$, for all $Q, Q' \in S$, for all $\tilde{S} \neq S$ and for all ν

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(Q,\mathcal{M})} q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) + \sum_{\alpha\in\mathcal{A}\setminus\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(Q,\mathcal{M})} q[Q, \tilde{S}, \alpha] \\
&= \sum_{\alpha\in\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(Q',\mathcal{M})} q[Q', \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, Q') + \sum_{\alpha\in\mathcal{A}\setminus\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(Q',\mathcal{M})} q[Q', \tilde{S}, \alpha] \ ,
\end{aligned}$$

Therefore, in light of what above explained, it holds that for any $S, \tilde{S}$ with $\tilde{S} \neq S$, and for any $P, P' \in S$

$$\sum_{\alpha \in \mathcal{A}} q[P, \tilde{S}, \alpha] \mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \sum_{\alpha \in \mathcal{A}} q[P', \tilde{S}, \alpha] \mathcal{F}_\alpha(\mathcal{M}, \nu, P') ,$$

for any ν, that is, it does not depend on the particular local state but only the corresponding block.

- It allows us to use Lemma 9 yielding that for any $P \in \mathcal{B}(\mathcal{M})$, for any $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P, \mathcal{M})$ and for any ν it holds

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}, P) .$$

  This point, instead, is used together with the sum decomposition in (7) and the Proposition 1 to infer that for any S, any $\tilde{S} \neq S$, any $P \in S$ and for any ν we have

$$\sum_{\alpha \in \mathcal{A}} q[P, \tilde{S}, \alpha] \mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \sum_{\alpha \in \mathcal{A}} q[P, \tilde{S}, \alpha] \mathcal{F}_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}, P) .$$

Now that all the proof ingredients have been provided, we can rewrite Equation (6) as follows

$$\begin{aligned}
\sum_{P \in S} f_P(\nu) = & \sum_{\tilde{S} \in \mathcal{P}/S} \left(\sum_{\alpha \in \mathcal{A}} q[P', S, \alpha] \mathcal{F}_\alpha(\mathcal{M}, \nu, P') \right) \sum_{P' \in \tilde{S}} \nu_{P'} \\
& - \left(\sum_{\tilde{S} \in \mathcal{P}/S} \left(\sum_{\alpha \in \mathcal{A}} q[P, \tilde{S}, \alpha] \mathcal{F}_\alpha(\mathcal{M}, \nu, P) \right) \right) \sum_{P \in S} \nu_P \\
= & \sum_{\tilde{S} \in \mathcal{P}/S} \left(\sum_{\alpha \in \mathcal{A}} q[P', S, \alpha] \mathcal{F}_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}, P') \right) \sum_{P' \in \tilde{S}} [\nu]^{\mathcal{P}}{}_{P'} \\
& - \left(\sum_{\tilde{S} \in \mathcal{P}/S} \left(\sum_{\alpha \in \mathcal{A}} q[P, \tilde{S}, \alpha] \mathcal{F}_\alpha(\mathcal{M}, [\nu]^{\mathcal{P}}, P) \right) \right) \sum_{P \in S} [\nu]^{\mathcal{P}}{}_P \\
= & \sum_{P \in S} f_P([\nu]^{\mathcal{P}}) ,
\end{aligned}$$

where we used that for any $S \in \mathcal{P}$ it holds $\sum_{P \in S} [\nu]^{\mathcal{P}}{}_P = \hat{\nu}_S = \sum_{P \in S} \nu_P$. □

C Results for DOL Characterisation (Proof of Theorem 2)

In this appendix we present all the results used to prove the characterisation of differential ordinary lumpability in terms of $\mathcal{CD}$-strong equivalence and $\mathcal{CD}$-context.

Theorem 5. *Let $\mathcal{M}$ be a well-posed FEPA model and $\mathcal{P}$ a differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M})$. Let $S \in \mathcal{P}$ and $P, Q \in S$. Then, for all $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(S, \mathcal{M})$ the two conditions below hold:*

$$q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha] \ , \qquad \textit{for all } \tilde{S} \in \mathcal{P} \ ,$$

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q) \ , \qquad \textit{for all } \nu \ .$$

Proof. Definition 11 guarantees that for any partition block S in $\mathcal{P}$, and any two components P, Q in S, we have

$$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) \tag{8}$$

for all $\tilde{S} \in \mathcal{P}$, $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(S, \mathcal{M})$, and for all ν. The well-posedness assumption of $\mathcal{M}$ allows us to use Lemma 5 on any two components in $\mathcal{B}(\mathcal{M})$. Lemma 5 tells us that, for any $\alpha \in \mathcal{A}$, there exists at least a population $\bar{\nu}_\alpha$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}_\alpha, P) = 1 = \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}_\alpha, Q)$. Given that Equation (8) holds for any ν, instantiating it with $\bar{\nu}_\alpha$ we obtain $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$, for any $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(S, \mathcal{M})$, and $\tilde{S} \in \mathcal{P}$, which concludes the proof of the first claim.

As regards the second claim, by applying to Equation (8) the result just shown, we have $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ for every $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(S, \mathcal{M})$ and for any ν, which concludes the proof of the second claim. This last step requires a more detailed explaination. We point out that for the actions in $\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(S, \mathcal{M})$ there exists at least a local state $P' \in S$ for which there exists at least a partition block $\tilde{S}$ such that $q[P', \tilde{S}, \alpha] > 0$. Exploiting the first claim we obtain $q[P', \tilde{S}, \alpha] = q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha] > 0$, allowing to rewrite Equation (8) as $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$, for all ν. □

The next corollary states that given a well-posed FEPA model $\mathcal{M}$ and a partition $\mathcal{P}$ of $\mathcal{B}(\mathcal{M})$, a necessary condition for the partition to be a differential ordinary lumpable partition is that all components within the same partition block are $\mathcal{CD}$–strong equivalent.

Corrolary 1. *Let $\mathcal{M}$ be a well-posed FEPA model and $\mathcal{P}$ a differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M})$. Then there exists a $\mathcal{CD}$–strong equivalence inducing the partition $\mathcal{P}$.*

Proof. Theorem 5 assures that for any $S \in \mathcal{P}$, for any $P, Q \in S$, and for any $\tilde{S}$, $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$ for any $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(S, \mathcal{M})$. This allows us to conclude that condition (ii) required by the definition of $\mathcal{CD}$–strong equivalence holds. Instead, the remaining conditions (i) and (iii) required by the definition of $\mathcal{CD}$–strong equivalence follow, respectively, from conditions (i) and (iii) of the definition of differential ordinary lumpability. □

We now move our attention towards the relation existing among the notion of differential ordinary lumpabilty and $\mathcal{CD}$-context.

Proposition 8. *Let $\mathcal{M}$ be a well-posed FEPA model, and $\hat{A}$ a set of actions. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be such that:*

i) $\mathcal{CD}^{\hat{A}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{A}}(Q, \mathcal{M})$,

ii) $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ *for all* $\alpha \in \mathcal{CD}^{\hat{A}}(P, \mathcal{M})$ *(as well as for all* $\alpha \in \mathcal{CD}^{\hat{A}}(Q, \mathcal{M})$*) and for all* ν.

Then, P and Q are in $\mathcal{CD}$–context with respect to $\hat{A}$.

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: This case follows from noticing that all local states in $\mathcal{B}(P)$ have empty current dependent set, and thus are in $\mathcal{CD}$–context in P with respect to any set of actions.

- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: We can have that either P and Q belong to the same sub-model $\mathcal{M}_i$, for $i \in \{1, 2\}$, or not. Without loss of generality, for the former case we assume $P, Q \in \mathcal{B}(\mathcal{M}_1)$, while for the latter $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$.

  We consider now the case $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$. By the assumption of the proposition we know that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$, and that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$, for any $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and for any ν. We want to show that this implies that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$. Due to the assumption $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$ and Definition 13, showing that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ reduces to show that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}) = \emptyset$. To prove this, let us assume, towards a contradiction, that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) \neq \emptyset$, and let α be in this set. We have to distinguish among two cases: $\alpha \in L$, $\alpha \notin L$.

  – $\alpha \in L$: By Definition 10 we have

  $$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \mathcal{F}_\alpha(\mathcal{M}_2, \nu, Q)\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_2, \nu)} .$$

  From the well-posedness of $\mathcal{M}$, together with Proposition 3, we can apply Lemma 4 firstly to $\mathcal{M}_1$ choosing a population $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P) = 1$, and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = K$, for a positive real K. We can then apply Lemma 4 to $\mathcal{M}_2$, choosing a population $\bar{\nu}_2$ such that $\mathcal{F}_\alpha(\mathcal{M}_2, \bar{\nu}_2, Q) = 1$, and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = K + 1$. Therefore, we have found a $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2)$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P) \neq \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, Q)$, obtaining a contradiction. Note that $r_\alpha(\mathcal{M}_i, \nu)$ depends only on the population functions assigned to the elements in $\mathcal{B}(\mathcal{M}_i)$.

  – $\alpha \notin L$: By Definition 10 we have

  $$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_2, \nu, Q) .$$

  The assumption that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and the information $\alpha \notin L$ implies $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ and thus $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$. We also remark that, taken a population function ν_1^0 for $\mathcal{M}_1$ which assign 0 population to all the local states in $\mathcal{B}(\mathcal{M}_1)$, Definition 3 implies $r_\alpha(\mathcal{M}_1, \nu_1^0) = 0$. We can thus apply Lemma 6 to $\mathcal{M}_1$, obtaining $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1^0, P) = 0$. Moreover, we know that $\mathcal{M}_2$ is a well-posed model, thus Proposition 4 assures the existence of a population function ν_2 for $\mathcal{M}_2$, such that $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_2, Q) > 0$. Hence, we have found a population function $\overline{\nu} = (\nu_1^0, \nu_2)$ for $\mathcal{M}$ such that $\mathcal{F}_\alpha(\mathcal{M}, \overline{\nu}, P) \neq \mathcal{F}_\alpha(\mathcal{M}, \overline{\nu}, Q)$, leading us to a contradiction.

  We consider now the case $P, Q \in \mathcal{B}(\mathcal{M}_1)$. In what follows we show that we can use the I.H. on $\mathcal{M}_1$ and thus infer that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}_1$. This information, together with the assumption that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$, assures that P and Q are also in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}$ (this can be deduced from Definition 13 and by noticing that the assumption $P, Q \in \mathcal{B}(\mathcal{M}_1)$ implies that if there exists an occurrence $\overline{\mathcal{M}} = \mathcal{M}'_1 \parallel_{L'}^{\mathcal{H}} \mathcal{M}'_2$ with $P \in \mathcal{B}(\mathcal{M}'_1)$ and $Q \in \mathcal{B}(\mathcal{M}'_2)$ within $\mathcal{M}$, it must be an occurrence within $\mathcal{M}_1$). In order to use the I.H. on $\mathcal{M}_1$, we have to prove that:

  a) $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$.

  b) $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ (as well as for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$) and for all ν.

We first show *a*). It is worth mentioning that assumption *i*) alone it is not enough to prove *a*). As a matter of fact, to prove this point we shall exploit assumption *i*) and *ii*) of the proposition, together with Proposition 5 and Lemma 7. More precisely, let us assume towards a contradiction that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \neq \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$. Without loss of generality, we may assume that there exists an action α such that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ and $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$. Moreover, by Proposition 5 and by the assumption $\mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$ we obtain that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$, and therefore $\alpha \in \mathcal{A}(P) \cap \hat{\mathcal{A}}$ and $\alpha \in \mathcal{A}(Q) \cap \hat{\mathcal{A}}$. From the above it follows that $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$, whereas $\alpha \notin \mathcal{D}(Q, \mathcal{M}_1)$. We now distinguish among two cases and show that, in either of them, we run into contradiction.

- $\alpha \notin L$: If $\alpha \notin L$ and $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$, the assumption $\mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$ implies that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$ and thus the contradiction.
- $\alpha \in L$: If $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ then, by Proposition 5, $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$. Hence, by assumption *ii*) of the proposition and by Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} ,$$

  for any ν. As mentioned before, we have that $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$ and $\alpha \notin \mathcal{D}(Q, \mathcal{M}_1)$. We can then apply Proposition 1 to infer that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, Q) = 1$ for any ν_1 on $\mathcal{M}_1$ (in fact $\alpha \notin \mathcal{D}(Q, \mathcal{M}_1)$), as well as Lemma 7 ($\alpha \in \mathcal{D}(P, \mathcal{M}_1)$, and $\mathcal{M}_1$ is well-posed due to the well-posedness assumption on $\mathcal{M}$ and Proposition 3) which assures for any K, ε, the existence of a function $\nu_{K,\varepsilon}$ on $\mathcal{M}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{K,\varepsilon}, P) = \varepsilon$ and $r_\alpha(\mathcal{M}_1, \nu_{K,\varepsilon}) = K$. The well-posedness assumption on $\mathcal{M}$ then also assures the existence of a population function ν_2 on $\mathcal{M}_2$ such that $r_\alpha(\mathcal{M}_2, \nu_2) > 0$. Therefore, choosing $K > 0$ and $\varepsilon < 1$, we have found a $\bar{\nu} = (\nu_{K,\varepsilon}, \nu_2)$ such that: $\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \bar{\nu})}{r_\alpha(\mathcal{M}_1, \bar{\nu})} > 0$, $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}, P) = \varepsilon$ and $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}, Q) = 1$ and thereby $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P) \neq \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, Q)$, which contradicts the assumption *ii*).

We address below the proof of *b*). Let $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ (note that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$). We now distinguish among two sub-cases: $\alpha \in L$, $\alpha \notin L$.

- $\alpha \in L$: By Proposition 5 we know that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$. Therefore, by assumption *ii*) of the proposition and by Definition 10 we have that for any ν it holds

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} .$$

  For all ν such that $\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} > 0$, we have $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$. For those ν such that $\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = 0$, instead, we can have that $r_\alpha(\mathcal{M}_1, \nu) = 0$ or $r_\alpha(\mathcal{M}_2, \nu) = 0$. In the case $r_\alpha(\mathcal{M}_1, \nu) = 0$, we can exploit Lemma 6 assuring that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) = 0$ (indeed, we know that if $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$ and $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$, then α belongs to $\mathcal{D}(P, \mathcal{M}_1)$ and $\mathcal{D}(Q, \mathcal{M}_1)$).

  For those ν such that $r_\alpha(\mathcal{M}_1, \nu) > 0$ and $r_\alpha(\mathcal{M}_2, \nu) = 0$ instead, we first recall a few facts; (a) ν can be seen as $\nu = (\nu_1, \nu_2)$, with ν_i defined for the model $\mathcal{M}_i$. (b) $\mathcal{F}_\alpha(\mathcal{M}_1, (\nu_1, \nu_2), P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P')$, for any ν_2, and any $P' \in \mathcal{B}(\mathcal{M}_1)$, and (c) $r_\alpha(\mathcal{M}_i, \nu) = r_\alpha(\mathcal{M}_i, \nu_i)$. With this results in mind, we exploit now the well-posedness of $\mathcal{M}$ (and the Proposition 3) which assures the existence of a population $\bar{\nu}_2$ such that $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) > 0$. Hence, being $r_\alpha(\mathcal{M}_1, \nu_1) > 0$ and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) > 0$, we would have that $\frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, (\nu_1, \bar{\nu}_2))}{r_\alpha(\mathcal{M}_1, (\nu_1, \bar{\nu}_2))} > 0$, implying $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, Q)$.

To sum up, we have just shown that for any $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \cap L$, $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for any ν.

– $\alpha \notin L$: By Proposition 5 we know that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$. Therefore, by assumption *ii*) of the proposition and by Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \ .$$

Hence, we have just shown that for any $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \setminus L$, $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for any ν.

Combining the results of the cases $\alpha \in L$ and $\alpha \notin L$, we have that for any $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1)$, $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for any ν. This having been proved, we can apply the I.H. to $\mathcal{M}_1$, ensuring that P and Q are in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}_1$. As mentioned before, this together with the fact that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$ implies that P and Q are also in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$ in $\mathcal{M}$, and concludes the proof.

□

Remark 1. *The previous proposition together with Theorem 5 implies that elements of a block of a DOLP $\mathcal{P}$ are in $\mathcal{CD}$–context with respect to $\mathcal{A}^{\mathcal{P}}_{\text{ext}}$. In fact, Proposition 8, instantiated with $\hat{\mathcal{A}} = \mathcal{A}^{\mathcal{P}}_{\text{ext}}$, can be applied to any pair of local states of any block of a DOLP by noticing that the assumption i) of the proposition follows directly from Definition 11, whilst assumption ii) is guaranteed by Theorem 5.*

The next proposition proves the reverse implication with respect to Proposition 8, i.e., that two local states in $\mathcal{CD}$-context with respect to a set of actions $\hat{\mathcal{A}}$ always receive the same influence from the rest of the model through $\hat{\mathcal{A}}$.

Proposition 9. *Let $\mathcal{M}$ be a well-posed FEPA model, and $\hat{\mathcal{A}}$ be a set of actions. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be in $\mathcal{CD}$–context with respect to $\hat{\mathcal{A}}$. Then, for all $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$ and for all ν, $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$.*

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: This case is trivial, as $\mathcal{CD}^{\hat{\mathcal{A}}}(P', \mathcal{M}) = \emptyset$ for any $P' \in \mathcal{B}(P)$.
- $\mathcal{M} = \mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2$: We can have that either P and Q belong to the same sub-model $\mathcal{M}_i$, for $i \in \{1, 2\}$, or not. Without loss of generality, for the former case we assume $P, Q \in \mathcal{B}(\mathcal{M}_1)$, while for the latter $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$.

  We now consider the case $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$. By the assumption that P and Q are in $\mathcal{CD}$–context wrt $\hat{\mathcal{A}}$ and by Definition 13, we know that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}) = \emptyset$ and the claim is therefore vacuously true. We now focus on the case $P, Q \in \mathcal{B}(\mathcal{M}_1)$. Let $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M})$ and consider the two sub-cases: $\alpha \in L$, $\alpha \notin L$.

  – $\alpha \in L$: By Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \ ,$$
$$\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \frac{r_\alpha(\mathcal{M}_1 \parallel^{\mathcal{H}}_L \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \ .$$

  What we are after is to prove that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ for any ν. By Proposition 7 we know that P and Q are in CD–context wrt $\hat{\mathcal{A}}$ in $\mathcal{M}_1$, and thus $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q, \mathcal{M}_1)$. Moreover, by Proposition 5 we also know that $\mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M}_1) \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(P, \mathcal{M})$

and $\mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M}_1) \subseteq \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M})$. If $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M}_1)$, we can exploit the I.H. to infer that $\mathcal{F}_\alpha(\mathcal{M}_1,\nu,P) = \mathcal{F}_\alpha(\mathcal{M}_1,\nu,Q)$ for every ν. Such an equality persists when one multiplies both terms for the same function $\frac{r_\alpha(\mathcal{M}_1\|^{\mathcal{H}}_L\mathcal{M}_2,\nu)}{r_\alpha(\mathcal{M}_1,\nu)}$ obtaining the claim. If $\alpha \notin \mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M}_1)$, instead, by the assumption that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M})$ we know that $\alpha \in \mathcal{A}(P)\cap\hat{\mathcal{A}}$ and $\alpha \in \mathcal{A}(Q)\cap\hat{\mathcal{A}}$, allowing us to conclude that $\alpha \notin \mathcal{D}(P,\mathcal{M}_1)$ and $\alpha \notin \mathcal{D}(Q,\mathcal{M}_1)$. Therefore, by Proposition 1 we have that $\mathcal{F}_\alpha(\mathcal{M}_1,\nu,P) = 1 = \mathcal{F}_\alpha(\mathcal{M}_1,\nu,Q)$ for every ν and thus the claim follows.

– $\alpha \notin L$: By Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M},\nu,P) = \mathcal{F}_\alpha(\mathcal{M}_1,\nu,P)\ ,$$
$$\mathcal{F}_\alpha(\mathcal{M},\nu,Q) = \mathcal{F}_\alpha(\mathcal{M}_1,\nu,Q)\ .$$

What we are after is to prove that $\mathcal{F}_\alpha(\mathcal{M},\nu,P) = \mathcal{F}_\alpha(\mathcal{M},\nu,Q)$ for any ν. By Proposition 7 we know that P and Q are in $\mathcal{CD}$–context wrt $\hat{\mathcal{A}}$ in $\mathcal{M}_1$. By the assumption $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M}) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M})$ and $\alpha \notin L$, by resorting to Definition 8 we can apply basic set theory and conclude that $\alpha \in \mathcal{CD}^{\hat{\mathcal{A}}}(P,\mathcal{M}_1) = \mathcal{CD}^{\hat{\mathcal{A}}}(Q,\mathcal{M}_1)$. In that case we use the I.H. to infer that $\mathcal{F}_\alpha(\mathcal{M}_1,\nu,P) = \mathcal{F}_\alpha(\mathcal{M}_1,\nu,Q)$ for any ν. The proof is then complete.

□

The next theorem gives the desired characterisation of differential ordinary lumpability, and concludes this appendix.

Theorem 2. *Let $\mathcal{M}$ be a well-posed FEPA model and $\mathcal{P}$ a partition of $\mathcal{B}(\mathcal{M})$. $\mathcal{P}$ is differential ordinary lumpable if and only if there exists a $\mathcal{CD}$–strong equivalence inducing the partition $\mathcal{P}$, and the local states of each block of $\mathcal{P}$ are in $\mathcal{CD}$-context with respect to $\mathcal{A}^{\mathcal{P}}_{\text{ext}}$.*

Proof. If $\mathcal{P}$ is a differential ordinary lumpable partition, then Corollary 1 guarantees that the partition $\mathcal{P}$ is induced by a $\mathcal{CD}$–strong equivalence. Moreover, Theorem 5 and the assumption that the partition $\mathcal{P}$ is a DOLP assure that the assumptions in Proposition 8 are met if instantiated with respect to the set of actions $\mathcal{A}^{\mathcal{P}}_{\text{ext}}$. We can therefore conclude that the local states of each block of $\mathcal{P}$ are in $\mathcal{CD}$–context with respect to $\mathcal{A}^{\mathcal{P}}_{\text{ext}}$.

As regards the opposite implication, the assumption that local states of each partition block $S \in \mathcal{P}$ are in $\mathcal{CD}$–context with respect to $\mathcal{A}^{\mathcal{P}}_{\text{ext}}$ allows us to use Proposition 9 to infer that for all $P,Q \in S$, for all $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P,\mathcal{M}) = \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P,\mathcal{M})$ and for all ν, $\mathcal{F}_\alpha(\mathcal{M},\nu,P) = \mathcal{F}_\alpha(\mathcal{M},\nu,Q)$. Furthermore, the assumption that the elements of the partition blocks of the same partition are also $\mathcal{CD}$–strong equivalent assures that $q[P,\tilde{S},\alpha] = q[Q,\tilde{S},\alpha]$ for all $\tilde{S} \in \mathcal{P}$, and for all $\alpha \in \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(P,\mathcal{M}) = \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{\text{ext}}}(Q,\mathcal{M})$. From the above, it follows that condition (ii) required by the definition of differential ordinary lumpability holds. On the other hand, conditions (i) and (iii) required by the definition of differential ordinary lumpability follow, respectively, from condition (i) and (iii) of the definition of $\mathcal{CD}$–strong equivalence. □

D Results regarding CoDOL (Proofs of Theorems 3,4)

In this appendix we provide the technical results regarding congruent differential ordinary lumpability.

We start by stating that congruent differential ordinary lumpability is a congruence with respect to parallel composition.

Theorem 3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two FEPA models, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be congruent differential ordinary lumpable partitions of $\mathcal{B}(\mathcal{M}_1)$ and $\mathcal{B}(\mathcal{M}_2)$, respectively. Then, the partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is a congruent differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2)$, for any $L \subseteq \mathcal{A}$.*

Proof. We first notice that for any interaction set L it holds $\mathcal{B}(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2) = \mathcal{B}(\mathcal{M}_1) \cup \mathcal{B}(\mathcal{M}_2)$. This, together with the disjointness assumption of the fluid atoms (yielding $\mathcal{B}(\mathcal{M}_1) \cap \mathcal{B}(\mathcal{M}_2) = \emptyset$), assures that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is a partition of $\mathcal{B}(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2)$. We are then left with proving that $\mathcal{P}$ is a congruent differential ordinary lumpable partition. The first condition, that is, for all $S \in \mathcal{P}$ and for all $P, Q \in S$ it holds $\mathcal{A}(P) = \mathcal{A}(Q)$, follows from noticing that any block of $\mathcal{P}$ is either a block of $\mathcal{P}_1$ or of $\mathcal{P}_2$, and they both satisfy (i) by assumptions. To prove condition (ii) instead, we need to show that for any $S, \tilde{S} \in \mathcal{P}$, for any $P, Q \in S$, for any $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and for any ν it holds that

$$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu, Q) \ ,$$

for any $L \subseteq \mathcal{A}$. Due to the disjointness assumption of the fluid atoms we can assume without loss of generality that $S, \tilde{S}$ are partition blocks of $\mathcal{P}_1$.

We need to distinguish between two cases, $\alpha \notin L$ and $\alpha \in L$.

- $\alpha \notin L$: The assumption that $S, \tilde{S} \in \mathcal{P}_1$ implies that $P, Q \in \mathcal{B}(\mathcal{M}_1)$. Hence, exploiting Definition 10 we need to show that for any ν it holds

  $$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \ .$$

  The assumption that $\mathcal{P}_1$ is a congruent differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M}_1)$ assures that

  $$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, Q) \ ,$$

  for any population function ν_1 for $\mathcal{M}_1$. This, together with the observation that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$ does not depend on the population function assigned to local states in $\mathcal{B}(\mathcal{M}_2)$, proves the claim.

- $\alpha \in L$: The assumption that $S, \tilde{S} \in \mathcal{P}_1$ implies that $P, Q \in \mathcal{B}(\mathcal{M}_1)$. Hence, exploiting Definition 10 we need to show that for any ν it holds

  $$[q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = q[Q, \tilde{S}, \alpha] \cdot$$
  $$\cdot \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \ .$$

  Similarly to the previous case, the assumption that $\mathcal{P}_1$ is a congruent differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M}_1)$ and the fact that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P)$ does not depend on the population function assigned to local states in $\mathcal{B}(\mathcal{M}_2)$ assures that

  $$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \ ,$$

  for any population function ν. Multiplying both sides for the same function $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)}$ the equality persists, and thus the claim follows.

□

The following proposition, given in Section 5, states that congruent differential ordinary lumpability (see Definition 14) implies differential ordinary lumpability (see Definition 11).

Proposition 2. *Let $\mathcal{M}$ be a well-posed FEPA model, and $\mathcal{P}$ be a partition of $\mathcal{B}(\mathcal{M})$. If $\mathcal{P}$ is a congruent differential ordinary lumpable partition, then it is also a differential ordinary lumpable partition.*

Proof. We need to show that for any $S \in \mathcal{P}$ and any two local states $P, Q \in S$ conditions $(i), (ii)$ and (iii) of Definition 11 are satisfied.

We first notice that if $\mathcal{P}$ is a congruent ordinary lumpable partition of $\mathcal{B}(\mathcal{M})$, then for any $S \in \mathcal{P}$ and any $P, Q \in S$ it holds that $\mathcal{CD}(P, \mathcal{M}) = \mathcal{CD}(Q, \mathcal{M})$. To see this, let us assume towards a contradiction, that $\mathcal{CD}(P, \mathcal{M}) \neq \mathcal{CD}(Q, \mathcal{M})$. Without loss of generality, we may then assume that there exists an $\alpha \in \mathcal{CD}(P, \mathcal{M})$ such that $\alpha \notin \mathcal{CD}(Q, \mathcal{M})$. Assumption (i) of CoDOL (i.e. $\mathcal{A}(P) = \mathcal{A}(Q)$) together with the definition of current dependent action set, would then imply that $\alpha \in \mathcal{D}(P, \mathcal{M})$ and $\alpha \notin \mathcal{D}(Q, \mathcal{M})$. Exploiting Proposition 1 for Q (guaranteeing that $\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) = 1$, for any ν) and Lemma 6 for P (assuring the existence of a population function $\bar{\nu}$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P) = 0$) condition (ii) of CoDOL, instantiated with $\bar{\nu}$ would read

$$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P) \overset{\substack{\alpha\in\\ \mathcal{D}(P,\mathcal{M})}}{=} 0 \overset{\text{CoDOL}}{=} q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, Q) \overset{\substack{\alpha\notin\\ \mathcal{D}(Q,\mathcal{M})}}{=} q[Q, \tilde{S}, \alpha] \ ,$$

for any $\tilde{S} \in \mathcal{P}$. This, however, contradicts the assumption $\alpha \in \mathcal{A}(Q)$, which implies the existence of at least a $\tilde{S} \in \mathcal{P}$ such that $q[Q, \tilde{S}, \alpha] > 0$. Therefore, any two elements P, Q of a block of a congruent differential ordinary lumpable partition also satisfy condition (i) of DOL, that is, $\mathcal{CD}(P, \mathcal{M}) \cap \mathcal{A}^{\mathcal{P}}_{ext} = \mathcal{CD}(Q, \mathcal{M}) \cap \mathcal{A}^{\mathcal{P}}_{ext}$ (we recall that the set $\mathcal{A}^{\mathcal{P}}_{ext}$ does not depend on the local state).

The fact that for any $S \in \mathcal{P}$ and any $P, Q \in S$ condition (ii) of DOL is also satisfied follows directly from condition (ii) of CoDOL by noticing that $\mathcal{CD}(P, \mathcal{M}) \cap \mathcal{A}^{\mathcal{P}}_{ext} \subseteq \mathcal{A}(P)$ for any P.

We are then left with proving that for any $S \in \mathcal{P}$ and any $P, Q \in S$ condition (iii) of DOL is satisfied. To see this, we remark that for any $S \in \mathcal{P}$ and any $P \in S$ the set $\mathcal{A} \setminus \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{ext}}(P, \mathcal{M})$ can be written as

$$\mathcal{A} \setminus \mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{ext}}(P, \mathcal{M}) = (\mathcal{A} \setminus \mathcal{A}(P)) \cup \left(\mathcal{A} \setminus \mathcal{A}^{\mathcal{P}}_{ext}\right) \cup (\mathcal{A} \setminus \mathcal{D}(P, \mathcal{M})) \ . \tag{9}$$

Moreover, for $\tilde{S} \neq S$, the only actions bringing a contribution to the summation $\sum_{\mathcal{A}\setminus\mathcal{CD}^{\mathcal{A}^{\mathcal{P}}_{ext}}(P,\mathcal{M})} q[P, \tilde{S}, \alpha]$, are those actions α such that $q[P, \tilde{S}, \alpha] > 0$. Exploiting Equation (9), the actions to be considered when verifying condition (iii) of DOL are only those actions α such that $\alpha \notin \mathcal{D}(P, \mathcal{M})$ and $\alpha \in \mathcal{A}(P)$. In fact, if $\alpha \in \mathcal{A} \setminus \mathcal{A}(P)$, then $q[P, \tilde{S}, \alpha] = 0$ for any $\tilde{S}$. Moreover, if $\alpha \in \mathcal{A} \setminus \mathcal{A}^{\mathcal{P}}_{ext}$, then $q[P, \tilde{S}, \alpha] = 0$ for any $\tilde{S}$ such that $P \notin \tilde{S}$ (otherwise α would belong to $\mathcal{A}^{\mathcal{P}}_{ext}$). For any $S \in \mathcal{P}$ and any $P, Q \in S$ the assumption that $\mathcal{P}$ is a congruent differential ordinary lumpable partition assures that $\mathcal{A}(P) = \mathcal{A}(Q)$ and, as shown above, that $\mathcal{CD}(P, \mathcal{M}) = \mathcal{CD}(Q, \mathcal{M})$. Thus, we can also infer that $\mathcal{A}(P) \setminus \mathcal{D}(P, \mathcal{M}) = \mathcal{A}(Q) \setminus \mathcal{D}(Q, \mathcal{M})$. Therefore, the set of actions bringing a non-zero contribution to the summation appearing in condition (iii) of DOL is the same for any $P, Q \in S$. For any action α in this set, condition (ii) of CoDOL implies that $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$ for any $\tilde{S}$ (and thus in particular any $\tilde{S} \neq S$). Indeed, for any $\alpha \in \mathcal{A}(P)$ and $\alpha \notin \mathcal{D}(P, \mathcal{M})$ Proposition 1 assures that $q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = q[P, \tilde{S}, \alpha]$ for any for any $\tilde{S}$ and for any ν. □

We now extend the characterisation results for differential ordinary lumpability presented in Appendix C to the congruent case. Although the results, as well as the proofs, are very similar in nature to those previously introduced for the DOL case, we provide them for the sake of completeness.

The following theorem is analogous to Theorem 5.

Theorem 6. *Let $\mathcal{M}$ be a well-posed FEPA model and $\mathcal{P}$ a congruent differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M})$. Let $S \in \mathcal{P}$ and $P, Q \in S$. Then, for all $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ the two conditions below hold:*

$$q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha] , \qquad \text{for all } \tilde{S} \in \mathcal{P} ,$$

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q) , \qquad \text{for all } \nu .$$

Proof. Definition 14 guarantees that for any partition block S in $\mathcal{P}$, and any two components P, Q in S, we have

$$q[P, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = q[Q, \tilde{S}, \alpha]\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) \tag{10}$$

for all $\tilde{S} \in \mathcal{P}$, $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$, and for all ν. The well-posedness assumption of $\mathcal{M}$ allows us to use Lemma 5 on any two components in $\mathcal{B}(\mathcal{M})$. Lemma 5 tells us that, for any $\alpha \in \mathcal{A}$, there exists at least a population $\bar{\nu}_\alpha$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}_\alpha, P) = 1 = \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}_\alpha, Q)$. Given that Equation (10) holds for any ν, instantiating it with $\bar{\nu}_\alpha$ we obtain $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$, for any $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$, and $\tilde{S} \in \mathcal{P}$, which concludes the proof of the first claim.

As regards the second claim, by applying to Equation (10) the result just shown, we have $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ for every $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and for any ν, which concludes the proof of the second claim. This last step requires a more detailed explaination. We point out that for an action $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ there exists at least a partition block $\tilde{S}$ such that $q[P, \tilde{S}, \alpha] > 0$. Exploiting the first claim we obtain $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha] > 0$, allowing to rewrite Equation (10) as $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$, for all ν. □

The next corollary states that given a well-posed FEPA model $\mathcal{M}$ and a partition $\mathcal{P}$ of $\mathcal{B}(\mathcal{M})$, a necessary condition for the partition to be congruent differential ordinary lumpable is that all components within the same partition block are strong equivalent.

Corrolary 2. *Let $\mathcal{M}$ be a well-posed FEPA model and $\mathcal{P}$ a congruent differential ordinary lumpable partition of $\mathcal{B}(\mathcal{M})$. Then there exists a strong equivalence inducing the partition $\mathcal{P}$.*

Proof. Theorem 6 assures that for any $S \in \mathcal{P}$, for any $P, Q \in S$, and for any $\tilde{S}$, $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$ for any $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$. This allows us to conclude that condition (ii) required by the definition of strong equivalence holds. Conditions (i) required by the definition of strong equivalence, instead, directly follows from conditions (i) of the definition of congruent differential ordinary lumpability. □

We now move our attention towards the relation existing between the notion of congruent differential ordinary lumpability and congruent $\mathcal{CD}$–context. Given a model $\mathcal{M}$, the next proposition states that the notion of congruent $\mathcal{CD}$–context is preserved while descending the syntax tree of the model.

Proposition 10. *Let $\mathcal{M}$ be a FEPA model. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be such that they are in congruent $\mathcal{CD}$–context in $\mathcal{M}$. Then for any sub-model $\mathcal{M}'$ of $\mathcal{M}$ such that $P, Q \in \mathcal{B}(\mathcal{M}')$, P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}'$ as well.*

Proof. We have to prove that either condition $i)$ or $ii)$ of Definition 16 hold for $\mathcal{M}'$. If $\overline{\mathcal{M}}$ does not occur in $\mathcal{M}$, then neither it occurs in $\mathcal{M}'$. If instead $\overline{\mathcal{M}}$ occurs in $\mathcal{M}$, then the fact that P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}$, and that $P, Q \in \mathcal{B}(\mathcal{M}')$ implies that $\overline{\mathcal{M}}$ must be an occurrence within $\mathcal{M}'$, with $\mathcal{CD}(P, \overline{\mathcal{M}}) = \mathcal{CD}(Q, \overline{\mathcal{M}}) = \emptyset$. □

Given two local states of a model, the presence of symmetries in their current actions, and in the influence they receive from the model through those actions, provides information on the structure of $\mathcal{M}$ for what concerns the interactions affecting the two local states. More specifically, given a model $\mathcal{M}$, a partition $\mathcal{P}$ of $\mathcal{B}(\mathcal{M})$, and two local states P, Q with the same current action set, if the model exerts the same influence on the rates with which P and Q perform their current actions regardless of the population functions, then we can infer that P and Q are in congruent $\mathcal{CD}$–context. The subsequent proposition formally addresses this issue.

Proposition 11. *Let $\mathcal{M}$ be a well-posed FEPA model. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be such that:*

i) $\mathcal{A}(P) = \mathcal{A}(Q)$,

ii) $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ *for all* $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ *and for all* ν.

Then, P and Q are in congruent $\mathcal{CD}$–context.

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: This case follows from noticing that P and Q satisfy condition (i) of Definition 16, as $P, Q \in \mathcal{B}(P)$, and thus it does not exist any occurrence $\overline{\mathcal{M}} = \mathcal{M}_1 \parallel_L \mathcal{M}_2$ within $\mathcal{M}$ with $P \in \mathcal{B}(\mathcal{M}_1)$, and $Q \in \mathcal{B}(\mathcal{M}_2)$ (or vice versa).

- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: We can have that either P and Q belong to the same sub-model $\mathcal{M}_i$, for $i \in \{1, 2\}$, or not. Without loss of generality, for the former case we assume $P, Q \in \mathcal{B}(\mathcal{M}_1)$, while for the latter $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$.

  We consider now the case $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$. By the assumption of the proposition we know that $\mathcal{A}(P) = \mathcal{A}(Q)$, and that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$, for any $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and for any ν. We want to show that this implies that P and Q are in congruent $\mathcal{CD}$–context. Due to the assumption $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$ and Definition 16, showing that P and Q are in congruent $\mathcal{CD}$–context reduces to show that $\mathcal{CD}(P, \mathcal{M}) = \mathcal{CD}(Q, \mathcal{M}) = \emptyset$. To prove this, let us assume, towards a contradiction, that $\mathcal{CD}(P, \mathcal{M}) \neq \emptyset$, and let α be in this set. We have to distinguish among two cases: $\alpha \in L$, $\alpha \notin L$.

  - $\alpha \in L$: By assumption $ii)$ of the proposition and by Definition 10 we have that for any ν it holds

    $$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \mathcal{F}_\alpha(\mathcal{M}_2, \nu, Q) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_2, \nu)} \; .$$

    From the well-posedness of $\mathcal{M}$, together with Proposition 3, we can apply Lemma 4 firstly to $\mathcal{M}_1$ choosing a population $\bar{\nu}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}_1, P) = 1$, and $r_\alpha(\mathcal{M}_1, \bar{\nu}_1) = K$, for a positive real K. We can then apply Lemma 4 to $\mathcal{M}_2$, choosing a population $\bar{\nu}_2$ such that $\mathcal{F}_\alpha(\mathcal{M}_2, \bar{\nu}_2, Q) = 1$, and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) = K + 1$. Therefore, we have found a $\bar{\nu} = (\bar{\nu}_1, \bar{\nu}_2)$ such that $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P) \neq \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, Q)$, obtaining a contradiction. Note that $r_\alpha(\mathcal{M}_i, \nu)$ depends only on the population functions assigned to the elements in $\mathcal{B}(\mathcal{M}_i)$.

  - $\alpha \notin L$: By Definition 10 we have

    $$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_2, \nu, Q) \; .$$

    The assumption that $\alpha \in \mathcal{CD}(P, \mathcal{M})$ and the information $\alpha \notin L$ implies $\alpha \in \mathcal{CD}(P, \mathcal{M}_1)$ and thus $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$. We also remark that, taken a population function ν_1^0 for $\mathcal{M}_1$ which assign 0 population to all the local states in $\mathcal{B}(\mathcal{M}_1)$, Definition 3 implies $r_\alpha(\mathcal{M}_1, \nu_1^0) = 0$. We can thus apply Lemma 6 to $\mathcal{M}_1$, obtaining $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1^0, P) = 0$. Moreover, we know that $\mathcal{M}_2$ is a well-posed model, thus Proposition 4 assures the existence of a population function ν_2 for $\mathcal{M}_2$, such that $\mathcal{F}_\alpha(\mathcal{M}_2, \nu_2, Q) > 0$. Hence, we have found a population function $\overline{\nu} = (\nu_1^0, \nu_2)$ for $\mathcal{M}$ such that $\mathcal{F}_\alpha(\mathcal{M}, \overline{\nu}, P) \neq \mathcal{F}_\alpha(\mathcal{M}, \overline{\nu}, Q)$, leading us to a contradiction.

We consider now the case $P, Q \in \mathcal{B}(\mathcal{M}_1)$. In what follows we show that we can use the I.H. on $\mathcal{M}_1$ and thus infer that P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}_1$. This information assures that P and Q are also in congruent $\mathcal{CD}$–context in $\mathcal{M}$ (this can be deduced from Definition 16 and by noticing that the assumption $P, Q \in \mathcal{B}(\mathcal{M}_1)$ implies that if there exists an occurrence $\overline{\mathcal{M}} = \mathcal{M}_1' \parallel_{L'}^{\mathcal{H}} \mathcal{M}_2'$ with $P \in \mathcal{B}(\mathcal{M}_1')$ and $Q \in \mathcal{B}(\mathcal{M}_2')$ within $\mathcal{M}$, it must be an occurrence within $\mathcal{M}_1$).

In order to use the I.H. on $\mathcal{M}_1$, we only have to prove that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for all $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and for all ν. Let $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$. We now distinguish among two sub-cases: $\alpha \in L$, $\alpha \notin L$.

– $\alpha \in L$: By assumption *ii*) of the proposition and by Definition 10 we have that for any ν it holds

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \,. \tag{11}$$

For all ν such that $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} > 0$, we have $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$. For those ν such that $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = 0$, instead, we can have that $r_\alpha(\mathcal{M}_1, \nu) = 0$ or $r_\alpha(\mathcal{M}_2, \nu) = 0$. For those ν such that $r_\alpha(\mathcal{M}_1, \nu) > 0$ and $r_\alpha(\mathcal{M}_2, \nu) = 0$, we first recall a few facts; (a) ν can be seen as $\nu = (\nu_1, \nu_2)$, with ν_i defined for the model $\mathcal{M}_i$. (b) $\mathcal{F}_\alpha(\mathcal{M}_1, (\nu_1, \nu_2), P') = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P')$, for any ν_2, and any $P' \in \mathcal{B}(\mathcal{M}_1)$, and (c) $r_\alpha(\mathcal{M}_i, \nu) = r_\alpha(\mathcal{M}_i, \nu_i)$. With this results in mind, we exploit now the well-posedness of $\mathcal{M}$ (and the Proposition 3) which assures the existence of a population $\bar{\nu}_2$ such that $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) > 0$. Hence, being $r_\alpha(\mathcal{M}_1, \nu_1) > 0$ and $r_\alpha(\mathcal{M}_2, \bar{\nu}_2) > 0$, we would have that $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, (\nu_1, \bar{\nu}_2))}{r_\alpha(\mathcal{M}_1, (\nu_1, \bar{\nu}_2))} > 0$, implying $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu_1, Q)$.

In the case $r_\alpha(\mathcal{M}_1, \nu) = 0$, instead, we distinguish among three cases:

(*i*) $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$ and $\alpha \in \mathcal{D}(Q, \mathcal{M}_1)$,

(*ii*) $\alpha \notin \mathcal{D}(P, \mathcal{M}_1)$ and $\alpha \notin \mathcal{D}(Q, \mathcal{M}_1)$,

(*iii*) $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$ and $\alpha \notin \mathcal{D}(Q, \mathcal{M}_1)$ (or vice versa).

In the (*i*) case, we can exploit Lemma 6 assuring that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) = 0$. In the case (*ii*) we can exploit Proposition 1 assuring that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu', P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu', Q) = 1$, for any ν'. As regards the case (*iii*), instead, we now show that this case contradicts the assumption *ii*) of the proposition, and hence does not have to be considered. In case (*iii*) we have that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu', Q) = 1$ for any possible ν', allowing us to rewrite Equation (11) as

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} = \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \,.$$

Given that $\alpha \in \mathcal{D}(P, \mathcal{M}_1)$ (and $\mathcal{M}_1$ is well-posed due to the well-posedness assumption and Proposition 3), we can apply Lemma 7, which assures that for any K, ε there exists a $\nu_{K,\varepsilon}$ on $\mathcal{M}_1$ such that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu_{K,\varepsilon}, P) = \varepsilon$ and $r_\alpha(\mathcal{M}_1, \nu_{K,\varepsilon}) = K$. The well-posedness assumption on $\mathcal{M}$ then also assures the existence of a population function ν_2 on $\mathcal{M}_2$ such that $r_\alpha(\mathcal{M}_2, \nu_2) > 0$. Therefore, choosing $K > 0$ and $\varepsilon < 1$, we have found a population $\bar{\nu} = (\nu_{K,\varepsilon}, \nu_2)$ such that: $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \bar{\nu})}{r_\alpha(\mathcal{M}_1, \bar{\nu})} > 0$, $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}, P) = \varepsilon$ and $\mathcal{F}_\alpha(\mathcal{M}_1, \bar{\nu}, Q) = 1$ and thereby $\mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, P) \neq \mathcal{F}_\alpha(\mathcal{M}, \bar{\nu}, Q)$, which contradicts the assumption *ii*).

To sum up, we have just shown that for any $\alpha \in \mathcal{A}(P) \cap L$, $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for any ν.

– $\alpha \notin L$: By assumption *ii*) of the proposition and by Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \ .$$

Hence, we have just shown that for any $\alpha \in \mathcal{A}(P) \setminus L$, $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for any ν.

Combining the results of the cases $\alpha \in L$ and $\alpha \notin L$, we have that for any $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$, $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for any ν. This having been proved, we can apply the I.H. to $\mathcal{M}_1$, ensuring that P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}_1$. As mentioned before, this implies that P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}$, and concludes the proof.

□

Remark 2. *The previous proposition together with Theorem 6 implies that elements of a block of a congruent differential ordinary lumpable partition are in congruent $\mathcal{CD}$–context. In fact, Proposition 11 can be applied to any pair of local states of any block of a CoDOLP by noticing that the assumption i) of the proposition follows directly from Definition 14, whilst assumption ii) is guaranteed by Theorem 6.*

The next proposition proves the reverse implication with respect to Proposition 11, i.e., that two local states in congruent $\mathcal{CD}$–context always receive the same influence from the rest of the model.

Proposition 12. *Let $\mathcal{M}$ be a well-posed FEPA model. Let $P, Q \in \mathcal{B}(\mathcal{M})$ be in congruent $\mathcal{CD}$–context. Then, for all $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and for all ν, $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$.*

Proof. The proof proceeds by structural induction on $\mathcal{M}$.

- $\mathcal{M} = P$: This case is trivial as, for any $P' \in \mathcal{B}(P)$, $\mathcal{F}_\alpha(P, \nu, P') = 1$ for any $\alpha \in \mathcal{A}$ and any ν.
- $\mathcal{M} = \mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2$: We can have that either P and Q belong to the same sub-model $\mathcal{M}_i$, for $i \in \{1, 2\}$, or not. Without loss of generality, for the former case we assume $P, Q \in \mathcal{B}(\mathcal{M}_1)$, while for the latter $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$.

  We now consider the case $P \in \mathcal{B}(\mathcal{M}_1)$, $Q \in \mathcal{B}(\mathcal{M}_2)$. By the assumption that P and Q are in congruent $\mathcal{CD}$–context and by Definition 16, we know that $\mathcal{CD}(P, \mathcal{M}) = \mathcal{CD}(Q, \mathcal{M}) = \emptyset$, i.e. $\mathcal{D}(P, \mathcal{M}) \cap \mathcal{A}(P) = \mathcal{D}(Q, \mathcal{M}) \cap \mathcal{A}(Q) = \emptyset$. We thus have that all actions that we are considering in this proposition (i.e. those in the set $\mathcal{A}(P) = \mathcal{A}(Q)$) are independent for both P and Q, and thus applying Proposition 1 we have $\mathcal{F}_\alpha(P, \mathcal{M}, \nu) = 1 = \mathcal{F}_\alpha(Q, \mathcal{M}, \nu)$ for any $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and any ν.

  We now focus on the case $P, Q \in \mathcal{B}(\mathcal{M}_1)$. Let $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and consider the two sub-cases: $\alpha \in L$, $\alpha \notin L$.

  – $\alpha \in L$: By Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \ ,$$
$$\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) \frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)} \ .$$

  What we are after is to prove that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ for any ν. By Proposition 10 we know that P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}_1$, allowing us to exploit the I.H. on $\mathcal{M}_1$ to infer that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for every ν. Such an equality persists when one multiplies both terms for the same function $\frac{r_\alpha(\mathcal{M}_1 \parallel_L^{\mathcal{H}} \mathcal{M}_2, \nu)}{r_\alpha(\mathcal{M}_1, \nu)}$ obtaining the claim.

– $\alpha \notin L$: By Definition 10 we have

$$\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) ,$$
$$\mathcal{F}_\alpha(\mathcal{M}, \nu, Q) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q) .$$

What we are after is to prove that $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$ for any ν. By Proposition 10 we know that P and Q are in congruent $\mathcal{CD}$–context in $\mathcal{M}_1$, allowing us to exploit the I.H. on $\mathcal{M}_1$ to infer that $\mathcal{F}_\alpha(\mathcal{M}_1, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}_1, \nu, Q)$ for every ν. The proof is then complete.

□

The next theorem gives the desired characterisation of congruent differential ordinary lumpability, and concludes this appendix.

Theorem 4. *Let $\mathcal{M}$ be a well-posed FEPA model and $\mathcal{P}$ a partition of $\mathcal{B}(\mathcal{M})$. $\mathcal{P}$ is congruent differential ordinary lumpable iff there exists a strong equivalence inducing $\mathcal{P}$, and the local states of each block of $\mathcal{P}$ are in congruent $\mathcal{CD}$-context.*

Proof. If $\mathcal{P}$ is a congruent differential ordinary lumpable partition, then Corollary 2 guarantees that the partition $\mathcal{P}$ is induced by a strong equivalence. Moreover, Theorem 6 and the assumption that the partition $\mathcal{P}$ is congruent differential ordinary lumpable assure that the assumptions in Proposition 11 are met. We can therefore conclude that the local states of each block of $\mathcal{P}$ are in congruent $\mathcal{CD}$–context.

As regards the opposite implication, condition (i) required by the definition of a congruent differential ordinary lumpability follows from both the definition of strong equivalence, as well as from that of congruent $\mathcal{CD}$–context. As regards condition (ii), the assumption that local states of each partition block $S \in \mathcal{P}$ are in congruent $\mathcal{CD}$–context allows us to use Proposition 12 to infer that for all $P, Q \in S$, for all $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$ and for all ν, $\mathcal{F}_\alpha(\mathcal{M}, \nu, P) = \mathcal{F}_\alpha(\mathcal{M}, \nu, Q)$. Furthermore, the assumption that the elements of the partition blocks of the same partition are also strong equivalent assures that $q[P, \tilde{S}, \alpha] = q[Q, \tilde{S}, \alpha]$ for all $\tilde{S} \in \mathcal{P}$, and for all $\alpha \in \mathcal{A}(P) = \mathcal{A}(Q)$. From the above, it follows that condition (ii) required by the definition of congruent differential ordinary lumpability holds. □